Two-dimensional array codes correcting 2×2 clusters of errors

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Abstract – Two–dimensional array codes that can correct two–dimensional clusters (or bursts) of errors are presented. New constructions of two-dimensional array codes correcting error clusters of size 2×2 are given.

1 Introduction

Clusters, patches or two–dimensional bursts of errors occur in many digital transmission, processing or storage systems, wherever data is formatted in two–dimensions. Two–dimensional array codes are very suitable for correcting cluster errors in such two-dimensional data structures (see [1]—[6] and the references therein).

In this paper we give new constructions of two-dimensional array codes correcting error clusters of size 2×2. Some constructed codes are better, in terms of excess redundancy, than the known classes of codes.

2 Preliminaries and Definitions

For integers $n_i, \ i = 1, 2$ we consider the set $V(n_1, n_2)$ of all binary two–dimensional $n_1 \times n_2$ arrays. A linear K–dimensional $(K \leq n_1 n_2)$ subspace $C(n_1, n_2)$ of the space $V(n_1, n_2)$ is called a linear binary two–dimensional array $[n_1 \times n_2, \ K]$ code of size (or area) $n_1 \times n_2$ with K information symbols and $r = n_1 n_2 - K$ parity-check symbols.

A $b_1 \times b_2$ cluster ($b_1 \leq n_1, \ b_2 \leq n_2$) is a nonzero $n_1 \times n_2$ array whose nonzero components are confined to a rectangle of size $b_1 \times b_2$. According to the Singleton–type bound [4], [7], the redundancy r required for a $b_1 \times b_2$–cluster–error–correcting code is

$$r \geq 2 b_1 b_2. \tag{1}$$

By analogy with one–dimensional codes ([8], p. 258) array codes that meet the bound (1) are said to be optimal. Optimal linear binary two–dimensional array codes correcting single error clusters of size $b_1 \times b_2$ for any integers b_1 and b_2 were given in [6]. These optimal linear binary two–dimensional array codes have relatively small sizes.

For $b_1 \times b_2$–cluster-error-correcting array codes whose sizes $n_1 \times n_2$ are much larger than $b_1 \times b_2$ the other criterion of the efficiency can be used. The criterion is based on the concept of the excess redundancy [2], [4]. We define the excess redundancy of the $b_1 \times b_2$–cluster–error–correcting array $[n_1 \times n_2,\ K]$ code C as

$$\tilde{r}_{n_1,n_2}(b_1,b_2) = r - \log_2 n_1 n_2, \tag{2}$$

where $r = n_1 n_2 - K$ is the redundancy of the code C. Now if $n = n_1 n_2$ and $n \to \infty$ we can define

$$\tilde{r}_C(b_1,b_2) = \lim_{n\to\infty} \tilde{r}_{n_1,n_2}(b_1,b_2), \tag{3}$$

if such limit exists. If this function is unbounded, we take $\tilde{r}_C(b_1,b_2) = \infty$.

Let $N(b_1,b_2)$ denote the number of distinct patterns of $b_1 \times b_2$ clusters. Then

$$\tilde{r}_C(b_1,b_2) \geq \log_2 N(b_1,b_2). \tag{4}$$

The inequality (4) follows immediately from the theorem 19 [2]. If $b_1 = b_2 = 2$, $N(2,2) = 10$ and

$$\tilde{r}_C(2,2) \geq \log_2 10. \tag{5}$$

3 Constructions of 2×2–cluster–error–correcting codes

Three different approaches to correction of 2×2 clusters of errors are presented.

The construction 1

Theorem 1 *Let n', n_1 and l be positive integers, $n' = n_1 l$, $l > 1$. If $v^{(1)} = (v_1^{(1)}, v_2^{(1)}, \ldots, v_{n'}^{(1)})$ and $v^{(2)} = (v_1^{(2)}, v_2^{(2)}, \ldots, v_{n'}^{(2)})$ are code words of a binary one-dimensional (n', k') code V correcting all 2–error–bursts $e = (e_1, e_2, \ldots, e_{n'})$ such that $e_i = e_{i+1} = 1$, $i \neq pn_1$, $p = 1, 2, \ldots, l-1$, $e_j = 0$, $j \neq i, i+1$, $1 \leq i, j \leq n'$, then the $n_1 \times 2l$ array*

$$c = \begin{bmatrix} v_1^{(1)} & v_1^{(2)} & \ldots & v_{(l-1)n_1+1}^{(1)} & v_{(l-1)n_1+1}^{(2)} \\ v_2^{(1)} & v_2^{(2)} & \ldots & v_{(l-1)n_1+2}^{(1)} & v_{(l-1)n_1+2}^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n_1}^{(1)} & v_{n_1}^{(2)} & \ldots & v_{ln_1}^{(1)} & v_{ln_1}^{(2)} \end{bmatrix} \tag{6}$$

is a code word of the binary two-dimensional array $[n_1 \times 2l, 2k]$ code C correcting single error clusters of size 2×2.

Example 1. (Lemma 1 [6]) Let α be a root of the polynomial $x^3 + x + 1$ over $GF(2)$ and V be the binary extended Hamming $[8,4]$ code with the parity–check matrix

$$H = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{7}$$

The code V corrects all 2–error–bursts except $e=(e_1,e_2,e_3,e_4,e_5,e_6,e_7,e_8)$, $e_4=e_5=1$, $e_j=0$, $j\neq 4,5$, $1\leq j\leq 8$. If $v^{(1)}=(v_1^{(1)},v_2^{(1)},v_3^{(1)},v_4^{(1)},v_5^{(1)},v_6^{(1)},v_7^{(1)},v_8^{(1)})$ and $v^{(2)}=(v_1^{(2)},v_2^{(2)},v_3^{(2)},$ $v_4^{(2)},v_5^{(2)},v_6^{(2)},v_7^{(2)},v_8^{(2)})$ are code words of the code V, then the 4×4 array

$$c=\begin{bmatrix} v_1^{(1)} & v_1^{(2)} & v_5^{(1)} & v_5^{(2)} \\ v_2^{(1)} & v_2^{(2)} & v_6^{(1)} & v_6^{(2)} \\ v_3^{(1)} & v_3^{(2)} & v_7^{(1)} & v_7^{(2)} \\ v_4^{(1)} & v_4^{(2)} & v_8^{(1)} & v_8^{(2)} \end{bmatrix} \tag{8}$$

is a code word of the optimal binary two-dimensional array $[4\times 4,8]$ code C correcting single error clusters of size 2×2.

The construction 2

This construction is based on the construction of array codes for cluster–error–correction [3] and direct–sum augmentation technique.

Let n_1,n_2 be positive integers, $n_1\geq 6$, $n_2\geq 8$ and

$$v=(v_{ij})=\begin{bmatrix} v_{0,0} & v_{0,1} & \cdots & v_{0,(n_2-1)} \\ v_{1,0} & v_{1,1} & \cdots & v_{1,(n_2-1)} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n_1-1,0} & v_{n_1-1,1} & \vdots & v_{n_1-1,n_2-1} \end{bmatrix} \tag{9}$$

be a code word of the binary two–dimensional array $[n_1\times n_2, n_1n_2-n_1-n_2+1]$ code V, that is the direct product of two simple parity–check codes V_1 and V_2. Thus, $v_i^{(1)}\in V_1$, $v_i^{(1)}=(v_{i,0},v_{i,1},\ldots,v_{i,n_2-1})$, $\sum_{j=0}^{n_2-1}v_{ij}=0$ and $v_j^{(2)}\in V_2$, $v_j^{(2)}=(v_{0,j},v_{1,j},\ldots,v_{n_1-1,j})$, $\sum_{i=0}^{n_1-1}v_{ij}=0$.

Let $u^{(1)}=(u_0^{(1)},u_1^{(1)},\ldots,u_{n_2-1}^{(1)})$ and $u^{(2)}=(u_0^{(2)},u_1^{(2)},\ldots,u_{n_1-1}^{(2)})$ be code words of binary one–dimensional 4–burst–error–correcting (n_1,k_1) and (n_2,k_2) codes U_1 and U_2, respectively, such that $\sum_{j=0}^{n_2-1}u_j^{(1)}=0$ and $\sum_{i=0}^{n_1-1}u_i^{(2)}=0$.

Let $t=(t_{ij})$ be the $n_1\times n_2$ array such that $t_{ij}=v_{ij}$ for $i=0,1,\ldots,n_1-2$, $j=0,1,\ldots,$ n_2-2, and $t_{n_1-1,j}=v_{n_1-1,j}+u_j^{(1)}$, $t_{i,n_2-1}=v_{i,n_2-1}+u_i^{(2)}$ for $i=0,1,\ldots,n_1-1$, $j=0,1,\ldots,n_2-1$.

Following [3] denote by ρ a rotation of a row $t_i=(t_{i,0},t_{i,1},\ldots,t_{i,n_2-1})$ to the right. So if $t_i=(01010011)$, then $\rho(t_i)=(10101001)$. Therefore, ρ^l denotes l rotations to the right and ρ^{-l} denotes l rotations to the left for $l\geq 0$. If $t_j=(t_{0,j},t_{1,j},\ldots,t_{n_1-1,j})^T$ is a column of $t=(t_{ij})$, then ρ^l denotes l rotations down. To each row t_i, $0\leq i\leq n_1-1$ of the array $t=(t_{ij})$ apply the rotation $\rho^{2<i>}$, where

$$< i >= \begin{cases} 1, & \text{if } i \text{ is odd,} \\ 0, & \text{if } i \text{ is even.} \end{cases}$$

Denote by $\hat{t}$ the $n_1 \times n_2$ array obtained this way. Next, apply to each column $\hat{t}_j,\ 0 \le j \le n_2 - 1$, of the array $\hat{t}$ the rotation $\rho^{2<j>}$. Denote by $z = (z_{ij})$ the resulting $n_1 \times n_2$ array.

Theorem 2 *The binary two-dimensional array code Z consisting of all words $z = (z_{ij})$ is the $[n_1 \times n_2,\ n_1 n_2 - n_1 - n_2 + k_1 + k_2 + 1]$ code correcting single error clusters of size 2×2.*

Example 2. Let $m_1, m_2 \ge 10$ be even integers and $U_1,\ U_2$ be binary one–dimensional shortened cyclic 4-burst-correcting codes of length $n_1 = 2^{m_1} - 2$ and $n_2 = 2^{m_2} - 2$ with $r_1 = m_1 + 3$ and $r_2 = m_2 + 3$ parity–check symbols, respectively (see the theorem 6 [9]). Then the binary two–dimensional array code Z is the $[(2^{m_1} - 2) \times (2^{m_2} - 2),\ 2^{m_1+m_2} - 2^{m_1+1} - 2^{m_2+1} - m_1 - m_2 - 1]$ code correcting single error clusters of size 2×2. The excess redundancy of the code Z is

$$\tilde{r}_{2^{m_1}-2,2^{m_2}-2}(2,2) = m_1 + m_2 + 5 - \log_2(2^{m_1} - 2)(2^{m_2} - 2).$$

It is easy to show that the excess redundancy of the class of the 2×2–cluster–error–correcting codes Z is equal to

$$\tilde{r}_Z(2,2) = 5.$$

The construction 3

For small n_1 (or n_2) we can use the construction of the theorem 4 [4] and one–dimensional b–burst–error–correcting codes with given properties to obtain two-dimensional array 2×2–cluster–error–correcting codes with good parameters. In particular, for $n_1 = 3$ we can construct 2×2–cluster-error–correcting codes with minimum excess redundancy.

Theorem 3 *Let n, n_1, n_2 and b be positive integers, $n = n_1 n_2,\ b = n_1 + 2$. If $v = (v_1, v_2, \ldots, v_n)$ is a code word of an one–dimensional b-burst–error–correcting (n,k) code V, then the $n_1 \times n_2$ array*

$$\hat{v} = \begin{bmatrix} v_1 & v_{n_1+1} & \cdots & v_{n_1(n_2-2)+1} & v_{n_1(n_2-1)+1} \\ v_2 & v_{n_1+2} & \cdots & v_{n_1(n_2-2)+2} & v_{n_1(n_2-1)+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n_1} & v_{2n_1} & \cdots & v_{n_1(n_2-1)} & v_{n_1 n_2} \end{bmatrix} \qquad (6)$$

is a code word of the binary two-dimensional array $[n_1 \times n_2, k]$ code $\hat{V}$ correcting single error clusters of size 2×2.

Example 3. The polynomial $g(x) = (1 + x^2 + x^3 + x^4)(1 + x + x^2 + x^3 + x^5 + x^9 + x^{10} + x^{13} + x^{14})$ generates the binary cyclic 5–burst–correcting $(2^{15} - 1, 2^{15} - 20)$ code V [9]. Using theorem 3 we can construct the two-dimensional array $[10922 \times 3, 32747]$ code $\hat{V}$ correcting single error clusters of size 2×2. The excess redundancy of the code $\hat{V}$ is

$$\tilde{r}_{10922,3}(2,2) = 19 - \log_2 32766.$$

Generalizing example 3 we can show that there exists the class of 2×2–cluster–error–correcting codes $\hat{V}$ with the excess redundancy

$$\tilde{r}_{\hat{V}}(2,2) = \lceil \log_2 10 \rceil,$$

where $\lceil a \rceil$ is the least integer more than or equal to a.

4 Conclusion

In this paper, we constructed new families of binary two-dimensional array code correcting single error clusters of size 2×2. An advantage of the constructed codes is illustrated by Tables 1 and 2. The described approaches can be used for constructing two-dimensional array codes correcting single error clusters of size $b_1 \times b_2$ for any integers b_1 and b_2.

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Size $n_1 \times n_2$	$n = n_1 n_2$	Redundancy r	Construction
4×4	16	8	Lemma 1 [6]
6×5	30	10	Theorem 1
3×16	48	11	Theorem 3
8×8	64	12	Construction B [5]
3×28	84	12	Theorem 3
3×96	288	13	Theorem 3
12×15	180	16	Two-dimensional Fire code [1]

Table 1: Parameters of 2×2–cluster–error–correcting $[n_1 \times n_2, K]$ codes of small size.

Class of codes	The excess redundancy $\tilde{r}_C(2,2)$
The burst identification and location (BIL) codes [2]	7
A class of FGZ codes (Theorem 4 [4])	$9 - log_2 9 \simeq 5,8$
A class of codes Z (Theorem 2)	5
A class of codes $\hat{V}$ (Theorem 3)	4

Table 2: The excess redundancy of 2×2–cluster–error–correcting $[n_1 \times n_2, K]$ codes of large size.

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