Weighted $\{\delta(p^3+1), \delta; 3, p^3\}$-minihypers and related linear codes meeting the Griesmer bound

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Abstract

Minihypers are equivalent to linear codes meeting the Griesmer bound. We classify all $\{\delta(p^3+1), \delta; 3, p^3\}$-minihypers, $\delta \leq 2p^2 - 4p$, $p = p_0^h \geq 9$, $h \geq 1$, for a prime number $p_0 \geq 7$. Such a minihyper is a sum of lines and of possibly one projected subgeometry $PG(5, p)$, or a sum of lines and a minihyper which is a projected subgeometry $PG(5, p)$ minus one line. When p is a square, also (possibly projected) Baer subgeometries $PG(3, p^{3/2})$ can occur. We will also discuss the general result on $\{\delta v_{\mu+1}, \delta v_\mu; t, q\}$-minihypers.

1 Weighted minihypers and linear codes meeting the Griesmer bound

Let $PG(t, q)$ be the t-dimensional projective space over $GF(q)$, the finite field of order q.

This abstract collects the results from [4, 3]. We refer to the abstract [2] for a similar threatment, although here, we will also describe the different types of projected $PG(5, p)$ on $PG(3, p^3)$. This will be done by discussing the space spanned by L, the line from which we project, and its conjugates.

A *linear* $[n, k, d; q]$ *code* C over the finite field $GF(q)$ of order q is a k-dimensional subspace of the n-dimensional vector space $V(n, q)$ over $GF(q)$, having minimum Hamming distance d.

From an economical point of view, it is interesting to use linear codes having a minimal length n for given k, d and q. The Griesmer bound states that if there exists a linear $[n, k, d; q]$ code for given values of k, d and q, then $n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil = g_q(k, d)$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x [8, 16].

We describe the link between weighted minihypers and linear codes meeting the Griesmer bound. These results were described in Hamada and Helleseth [10].

Definition 1.1 (Hamada and Tamari [12]) *An* $\{f, m; N, q\}$-minihyper *is a pair* (F, w)*, where* F *is a subset of the point set of* $PG(N, q)$ *and* w *is a weight function* $w : PG(N, q) \to \mathbb{N} : x \mapsto w(x)$*, satisfying*
(1) $w(x) > 0 \Leftrightarrow x \in F$,
(2) $\sum_{x \in F} w(x) = f$*, and*
(3) $min(\sum_{x \in H} w(x) \mid H \in \mathcal{H}) = m$*; where* $\mathcal{H}$ *denotes the set of hyperplanes of* $PG(N, q)$.

In the case that w is a mapping onto $\{0,1\}$, the minihyper (F,w) can be identified with the set F and is simply denoted by F.

The excess e *of a minihyper (F,w) is the number $\sum_{x\in F}(w(x)\Leftrightarrow 1)$.*

Suppose there exists a linear $[n,k,d;q]$-code meeting the Griesmer bound ($d \geq 1, k \geq 3$), then we can write d in an unique way as $d = \theta q^{k-1} \Leftrightarrow \sum_{i=0}^{k-2} \epsilon_i q^{\lambda_i}$ such that $\theta \geq 1$ and $0 \leq \epsilon_i < q$.

Using this expression for d, the Griesmer bound for an $[n,k,d;q]$ code can be expressed as: $n \geq \theta v_k \Leftrightarrow \sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i+1}$ where $v_l = (q^l \Leftrightarrow 1)/(q \Leftrightarrow 1)$, for any integer $l \geq 0$.

Let $\bar{E}(t,q)$ denote the set of all ordered tuples $(\zeta_0,\ldots,\zeta_{t-1})$ of integers ζ_i such that $(\zeta_0,\ldots,\zeta_{t-1}) \neq (0,\ldots,0)$ and either: (a) $0 \leq \zeta_0 \leq q \Leftrightarrow 1, 0 \leq \zeta_1 \leq q \Leftrightarrow 1, \ldots, 0 \leq \zeta_{t-1} \leq q \Leftrightarrow 1$, or (b) $\zeta_0 = q, 0 \leq \zeta_1 \leq q \Leftrightarrow 1, \ldots, 0 \leq \zeta_{t-1} \leq q \Leftrightarrow 1$, or (c) $\zeta_0 = \ldots = \zeta_{\lambda-1} = 0$, $\zeta_\lambda = q$, $0 \leq \zeta_{\lambda+1} \leq q \Leftrightarrow 1, \ldots, 0 \leq \zeta_{t-1} \leq q \Leftrightarrow 1$ for some integer $\lambda \in \{1,\ldots,t \Leftrightarrow 1\}$.

From now on, we suppose that $(\epsilon_0,\ldots,\epsilon_{k-2})$ belongs to $\bar{E}(k \Leftrightarrow 1, q)$.

Hamada and Helleseth [10, 11] showed that there is a one-to-one correspondence between the set of all non-equivalent $[n,k,d;q]$ codes meeting the Griesmer bound and the set of all projectively distinct $\{\sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i+1}, \sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i}; k \Leftrightarrow 1, q\}$-minihypers (F,w), such that $1 \leq w(p) \leq \theta$ for every point $p \in F$.

More precisely, the link is described in the following way. Let $G = (g_1 \cdots g_n)$ be a generator matrix for a linear $[n,k,d;q]$ code, meeting the Griesmer bound. We look at a column of G as being the coordinates of a point in $PG(k \Leftrightarrow 1, q)$. Let the point set of $PG(k \Leftrightarrow 1, q)$ be $\{s_1,\ldots,s_{v_k}\}$. Let $m_i(G)$ denote the number of columns in G defining s_i. Let $m(G)$ be the maximum value in $\{m_i(G) \mid i = 1,2,\ldots,v_k\}$. Then $\theta = m(G)$ is uniquely determined by the code C and we call it the *maximum multiplicity* of the code. Define the weight function $w : PG(k \Leftrightarrow 1, q) \rightarrow \mathbb{N}$ as $w(s_i) = \theta \Leftrightarrow m_i(G)$, $i = 1,2,\ldots,v_k$. Let $F = \{s_i \in PG(k \Leftrightarrow 1, q) \mid w(s_i) > 0\}$, then (F,w) is a $\{\sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i+1}, \sum_{i=0}^{k-2} \epsilon_i v_{\lambda_i}; k \Leftrightarrow 1, q\}$-minihyper with weight function w.

2 A particular class of minihypers

Minihypers have many applications in finite geometries [1, 5, 6, 7]. A class of minihypers which is crucial in the study of *maximal partial t-spreads* and *minimal t-covers* in finite projective spaces $PG(n,q)$, where $(t+1)|(n+1)$, is the class of $\{\delta v_{t+1}, \delta v_t; n, q\}$-minihypers.

A further reason to study this particular class of minihypers is that for this class, there is a duality principle; indeed, as will be explained later on more in detail, for a $\{\delta v_2, \delta v_1; 3, q\}$-minihyper (F,w), the blocking planes through a point of (F,w) form a dual blocking set in the quotient geometry of this point.

We improve the results of [6]. By using the recent results on the classification of the smallest minimal blocking sets B in $PG(2,p^3)$, new classification results on $\{\delta v_{t+1}, \delta v_t; n, p^3\}$-minihypers are obtained.

We present the results for $\{\delta v_2, \delta v_1; 3, p^3\}$-minihypers, and refer to [4] where the other cases for (t,n) are discussed.

The easiest way to construct weighted minihypers is to construct a *sum* of certain geometrical objects.

Consider a number of geometrical objects, such as subspaces $PG(d, q = p^h)$ of $PG(n, q = p^h)$, subgeometries $PG(d, p^t)$ of $PG(n, q = p^h)$, where $t|h$, and even projected subgeometries $PG(d, p^t)$ in $PG(n, q = p^h)$, where $t|h$. In the first two cases, a point of respectively $PG(d,q)$

or $PG(d,p^t)$ has weight one, while all the other points not belonging to respectively $PG(d,q)$ or $PG(d,p^t)$ have weight zero. In the latter case, let Π be a projected $PG(d,p^t)$. The *weight* of a point $s \in \Pi$ of the projected subgeometry Π is the number of points s' of $PG(d,p^t)$ that are projected onto s; all other points s not belonging to Π have weight zero.

Then the *sum* of these subspaces and (projected) subgeometries is the weighted set (F,w), where the weight $w(s)$ of a point s of (F,w) is the sum of all the weights of s in the subspaces and (projected) subgeometries of (F,w).

We will characterize the $\{\delta(p^3+1),\delta;3,p^3\}$-minihypers, δ small, with excess $e \le p^3$, as being either: (1) a sum of lines, (projected) subgeometries $PG(3,p^{3/2})$ when p is square, and of at most one projected $PG(5,p)$, or (2) the sum of lines, (projected) subgeometries $PG(3,p^{3/2})$ when p is square, and one $\{(p^2+p)(p^3+1),p^2+p;3,p^3\}$-minihyper which is a projected $PG(5,p)$ minus one line.

The crucial substructures that are used in the characterizations of the minihypers are so-called *blocking sets*.

Definition 2.1 *A* blocking set *of $PG(2,q)$ is a set of points intersecting every line of $PG(2,q)$ in at least one point.*

A blocking set is called minimal *when no proper subset of it is still a blocking set; and we call a blocking set* non-trivial *when it contains no line.*

A blocking set of $PG(2,q)$ is called small *when it has less than $3(q+1)/2$ points.*

If $q=p^h$, p prime, we call the exponent *e of the minimal blocking set B the maximal integer e such that every line intersects B in 1 modulo p^e points.*

From a result of Szőnyi [17], it follows that $e \ge 1$ for every small non-trivial minimal blocking set in $PG(2,q)$.

A plane intersecting a minihyper (F,w) of $PG(3,q)$ in a blocking set will be called a *blocking plane* of (F,w).

Crucial in our classification results are the recent classification results on non-trivial minimal blocking sets in $PG(2,p^3)$.

3 Known results on blocking sets

Theorem 3.1 (Polverino [13, 14], Polverino and Storme [15]) *The smallest minimal blocking sets in $PG(2,p^3)$, $p=p_0^h$, p_0 prime, $p_0 \ge 7$, with exponent $e \ge h$, are:*

(1) *a line,*
(2) *a Baer subplane of cardinality $p^3+p^{3/2}+1$, when p is a square,*
(3) *a set B of cardinality p^3+p^2+1, equivalent to*

$$\{(x,T(x),1) \mid x \in GF(p^3)\} \cup \{(x,T(x),0) \mid x \in GF(p^3)\setminus\{0\}\},$$

with T the trace function from $GF(p^3)$ to $GF(p)$, i.e., $T: GF(p^3) \to GF(p): x \mapsto x+x^p+x^{p^2}$.

A line intersects this blocking set B in $1, p+1$ or p^2+1 points. The last type of intersection with a line will be called a (p^2+1)-set.

(4) *a set B of cardinality p^3+p^2+p+1, equivalent to*

$$\{(x,x^p,1) \mid x \in GF(p^3)\} \cup \{(x,x^p,0) \mid x \in GF(p^3)\setminus\{0\}\}.$$

A line intersects B in $1, p+1$ or p^2+p+1 points.

The last type of intersection with a line will be called a (p^2+p+1)-set.

Remark 3.2 These two latter blocking sets (3) and (4) are also characterized as being a projected $PG(3,p)$ in the plane $PG(2,p^3)$. Namely, embed $PG(2,p^3)$ in a 3-dimensional space $PG(3,p^3)$. Consider a subgeometry $PG(3,p)$ of $PG(3,p^3)$ and a point r not belonging to this subgeometry $PG(3,p)$ and not belonging to the plane $PG(2,p^3)$.

Project $PG(3,p)$ from r onto $PG(2,p^3)$.

If the point r belongs to a line of the subgeometry $PG(3,p)$, then this $PG(3,p)$ is projected onto the blocking set of size p^3+p^2+1; otherwise we obtain the blocking set of size p^3+p^2+p+1.

Important in our techniques is the following result on plane intersections of a minihyper (F,w) in $PG(3,q)$.

Theorem 3.3 (Hamada and Helleseth [9]) *Let (F,w) be a $\{\delta(q+1),\delta;t,q\}$-minihyper where $t \geq 3$, $\delta \leq 2p^2$.*

Then a plane of $PG(t,q)$ is either contained in (F,w) or intersects (F,w) in an $\{m_0+m_1(q+1),m_1;2,q\}$-minihyper with $m_0+m_1=\delta$.

For a plane intersecting a minihyper (F,w) in an $\{m_0+m_1(q+1),m_1;2,q\}$-minihyper, we will call m_1 the multiplicity of that plane with respect to the minihyper (F,w).

Lemma 3.4 (Govaerts and Storme [6]) *A point of a $\{\delta(q+1),\delta;3,q\}$-minihyper (F,w) having weight one is contained in exactly $q+\delta$ planes with respect to (F,w), counted with multiplicities.*
A point having weight zero with respect to a $\{\delta(q+1),\delta;3,q\}$-minihyper (F,w) is contained in exactly δ planes with respect to (F,w), counted with multiplicities.

Lemma 3.5 (Govaerts and Storme [6]) *A line L contains α points of a $\{\delta(q+1),\delta;3,q\}$-minihyper (F,w) in $PG(3,q)$ if and only if there are exactly α planes with respect to (F,w) through L.*

4 Examples

The main problem in the classification results on $\{\delta(p^3+1),\delta;3,p^3\}$-minihypers F that will be presented is that such minihypers might contain projected subgeometries $PG(5,p) \equiv \Omega$.

We now give the detailed description of the different types of points s in a projected subgeometry $PG(5,p) \equiv \Omega$ in $PG(3,p^3)$, and of the planes of $PG(3,p^3)$ passing through s which share a projected subgeometry $PG(3,p)$ with Ω.

Consider a subgeometry $\Delta = PG(5,p)$ naturally embedded in $PG(5,p^3)$. Let L be a line of $PG(5,p^3)$ skew to Δ. Then the line L has two conjugate lines with respect to Δ. We will always denote these conjugate lines by L^p and L^{p^2}.

Case 1. Suppose Ω is the projection of a $PG(5,p) \equiv \Delta$ from a line L with $\dim\langle L, L^p, L^{p^2}\rangle = 5$.

Then every projected point s in Ω has weight one. Every point $s \in \Omega$ lies on exactly one (p^2+p+1)-set, on $p^4+p^3+p^2$ $(p+1)$-secants, and lies in p^3+p^2+p+1 planes of $PG(3,p^3)$ sharing a minimal 1-fold blocking set of size p^3+p^2+p+1 with Ω.

This is proven in the following way. Let s be the projection of the point s' of Δ. The planes $\langle r, r^p, r^{p^2}\rangle \cap \Delta$, $r \in L$, induce a regular 2-spread in Δ, i.e., a partitioning of the point set of Δ into planes. The planes of this regular 2-spread are projected onto (p^2+p+1)-sets of Ω; thus implying that s lies on exactly one (p^2+p+1)-set of Ω. Through such a plane $\langle r, r^p, r^{p^2}\rangle \cap \Delta$, $r \in L$, there pass p^2+p+1 3-spaces of Δ which are projected onto planar minimal blocking sets of size p^3+p^2+p+1.

And, similarly, if one considers a 3-space of Δ defined by s' and a plane $\langle r', r'^p, r'^{p^2}\rangle \cap \Delta$, $r' \in L$, with $s' \notin \langle r', r'^p, r'^{p^2}\rangle \cap \Delta$; also such a 3-space of Δ is projected onto a planar minimal blocking set of size p^3+p^2+p+1.

This shows that s lies in total on p^3+p^2+p+1 planes of $PG(3,p^3)$ sharing a minimal 1-fold blocking set of size p^3+p^2+p+1 with Ω.

In general, a plane of $PG(3,p^3)$ intersects Ω in either a $PG(2,p)$, a (p^2+p+1)-set, or in a minimal blocking set of size p^3+p^2+p+1.

Case 2. Suppose Ω is the projection of a $PG(5,p) \equiv \Delta$ from a line L with $\dim\langle L, L^p, L^{p^2}\rangle = 4$.

Then the 4-dimensional space $\langle L, L^p, L^{p^2}\rangle \cap \Delta$ is called the *special* 4-space of Δ, and similarly, its projection is called the *special* projected 4-space of Ω. We will denote this special 4-space $\langle L, L^p, L^{p^2}\rangle \cap \Delta$ by $\mathcal{P}$.

Then for exactly one point r of L, $\dim\langle r, r^p, r^{p^2}\rangle = 1$. This line $M = \langle r, r^p, r^{p^2}\rangle$ is projected from L onto a point of Ω of weight $p+1$. The other p^3 points r of L satisfy $\dim\langle r, r^p, r^{p^2}\rangle = 2$. The latter planes $\langle r, r^p, r^{p^2}\rangle \cap \Delta$ are projected onto (p^2+p+1)-sets of $PG(3,p^3)$.

Let s be the point of Ω of weight $p+1$. Every plane π of Δ passing through M and not lying in $\mathcal{P}$ is projected from L onto a (p^2+1)-set with special point s. Each such plane π lies in p^2+p+1 solids of Δ which are projected onto planar minimal blocking sets of size p^3+p^2+1; thus implying that s lies in $p^4+p^3+p^2$ planes of $PG(3,p^3)$ sharing a 1-fold blocking set of size p^3+p^2+1 with Ω.

Let s be a point of Ω different from the point of weight $p+1$ and not lying in the special 4-space $\mathcal{P}$ of Ω. Assume s is the projection of $s' \in \Delta$. Then each solid $\langle r, r^p, r^{p^2}, s'\rangle$, with $r \in L \setminus M$, is projected onto a planar minimal blocking set of size p^3+p^2+p+1; hence, s lies in p^3 such planes. And every solid of Δ passing through M and s' is projected onto a planar minimal blocking set of size p^3+p^2+1 passing through s; thus giving p^2+p+1 extra planes through s intersecting Ω in a projected $PG(3,p)$.

Let s be a point of weight one of Ω which is the projection of a point s' of $\mathcal{P}$. Then the plane $\langle M, s'\rangle$ lies in p^2 3-spaces of Δ not contained in $\mathcal{P}$ which are projected onto planar blocking sets of size p^3+p^2+1 through s.

Case 3. Suppose Ω is the projection of a $PG(5,p) \equiv \Delta$ from a line L with $\dim\langle L, L^p, L^{p^2}\rangle = 3$.

Let $\mathcal{P} = \langle L, L^p, L^{p^2}\rangle \cap \Delta$.

Every plane α through L in $\langle L, L^p, L^{p^2}\rangle$ has two conjugate planes α^p, α^{p^2} with respect to Δ, and these three planes intersect in at least one point of $\mathcal{P}$. Hence every plane through L in $\langle L, L^p, L^{p^2}\rangle$ contains at least one point of $\mathcal{P}$. Then we call the 3-dimensional space $\mathcal{P}$

the *special* 3-space of Δ, and its projection will always be denoted by the line N. There are $p+1$ skew lines $L_1, \dots, L_{p+1}$ in $\mathcal{P}$ which are projected onto points of weight $p+1$, and the remaining $p^3 \Leftrightarrow p$ points of $\mathcal{P}$ are projected onto points of weight one of the line N.

A point s' of $\Delta \setminus \mathcal{P}$ is projected onto a point s lying on $p+1$ (p^2+1)-secants, which are the projections of $\langle s', L_i \rangle \cap \Delta$, $i = 1, \dots, p+1$. Each such (p^2+1)-secant through s lies in p^2 planes of $PG(3,p^3)$ containing a projected $PG(3,p)$ of Δ, which is a minimal blocking set of size p^3+p^2+1; hence, s' lies in p^3+p^2 such planes. Considering these $PG(3,p)$'s in Δ; these are the $PG(3,p)$'s through a plane $\langle s', L_i \rangle$ only intersecting $\mathcal{P}$ in L_i.

Furthermore, $\mathcal{P}$ is projected on the line N through which there are $p+1$ planes of $PG(3,p^3)$ containing $p^4+p^3+p^2+p+1$ projected points of Δ. The other planes through N contain p^3+p^2+p+1 projected weighted points; these all lie on N.

Hence, this projection forms a $\{(p^2+p+1)(p^3+1), p^2+p+1; 3, p^3\}$-minihyper containing the line N. Reducing the weight of every point on N by one yields a $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper.

Case 4. Suppose Ω is the projection of a $PG(5,p) \equiv \Delta$ from a line L with $\dim\langle L, L^p, L^{p^2}\rangle = 2$.

Then this projection is a cone of p^2+p+1 lines; the vertex of the cone is a common point having weight p^2+p+1 arising from the projection of the points of the plane $\langle L, L^p, L^{p^2}\rangle \cap \Delta$, and the base of the cone is a subplane $PG(2,p)$.

5 The classification result on minihypers

We now start the description of the arguments leading to the classification result of Theorem 5.7. We assume that (F,w) is a weighted minihyper satisfying the conditions of Theorem 5.7.

The first result shows that we can assume that (F,w) does not contain any lines.

Theorem 5.1 *If (F,w) contains a line, then we can delete this line from (F,w) to obtain a $\{(\delta \Leftrightarrow 1)(q+1), \delta \Leftrightarrow 1; 3, q\}$-minihyper.*

Now we describe this duality property that is valid for the $\{\delta(p^3+1), \delta; 3, p^3\}$-minihypers, and that was mentioned at the beginning of Section 2.

Consider a point r of (F,w) with weight one. If we consider the planes of $PG(3,p^3)$ through r, according to their multiplicities with respect to (F,w), then they form a dual non-trivial blocking set of size $p^3+\delta$ in the quotient geometry of r.

We will describe this quotient geometry by means of a plane π_r skew to r, and denote the dual blocking set of blocking planes in π_r by ${B_r}^D$.

This dual blocking set contains a dual minimal blocking set E. By the classification results of Polverino and Storme (Theorem 3.1), there are three possibilities for this dual minimal blocking set E. We discuss the three possibilities separately.

Theorem 5.2 *If E is a Baer subplane, then (F,w) contains a $PG(3,p^{3/2})$ through r.*

Theorem 5.3 *It is impossible that E has size p^3+p^2+1.*

So only the dual of the minimal blocking set of size p^3+p^2+p+1 remains as possibility for E (Theorem 3.1). This latter blocking set intersects exactly one line in p^2+p+1 points. Dualizing this property, the following result is obtained.

Remark 5.4 There is a special point $s_0 \in \pi_r$, contained in p^2+p+1 lines of E.

We now use these p^2+p+1 lines of E through r. These lines of E through r define p^2+p+1 blocking planes of (F,w) through r. We use the intersections of these planes with (F,w) to construct a projected subgeometry $PG(5,p)$ completely contained in (F,w).

Lemma 5.5 *There are at least three planes α, β, γ defined by r and a line of E, which satisfy the following properties:*
(1) *they intersect (F,w) in an 1-fold blocking set,*
(2) *these 1-fold blocking sets $\alpha \cap (F,w), \beta \cap (F,w), \gamma \cap (F,w)$ contain minimal 1-fold blocking sets E_1, E_2, E_3 which are projected subgeometries $PG(3,p)$ sharing the same (p^2+1)- or (p^2+p+1)-set on rs_0, and*
(3) *these projected subgeometries E_1, E_2, E_3 define a projected subgeometry $\Omega \equiv PG(5,p)$ of $PG(3,p^3)$.*

Using the other planes through r defined by a line of E, it is possible to prove that this latter projected subgeometry Ω lies completely in (F,w).

Theorem 5.6 *The projected subgeometry $PG(5,p) \equiv \Omega$ is contained in (F,w).*

We have now discussed all the possibilities. We have obtained the following characterization result on minihypers.

Theorem 5.7 (Ferret and Storme [3]) *A $\{\delta(p^3+1), \delta; 3, p^3\}$-minihyper, $p = p_0^h$, p_0 prime, $h \geq 1$, $p_0 \geq 7$, $p \geq 9$, $\delta \leq 2p^2 \Leftrightarrow 4p$, and with excess $e \leq p^3$, is either:*
(1) *a sum of lines, (projected) $PG(3,p^{3/2})$'s, and at most one projected $PG(5,p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle \geq 3$,*
(2) *a sum of lines, (projected) $PG(3,p^{3/2})$'s, and a $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $\Omega \setminus N$, where Ω is a $PG(5,p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, and where N is the line contained in Ω.*

6 The general result

We first of all extended the result on the $\{\delta(p^3+1), \delta; 3, p^3\}$-minihypers of Theorem 5.7 to $\{\delta(p^3+1), \delta; N, p^3\}$-minihypers, with $N \geq 4$.

Theorem 6.1 (Ferret and Storme [4]) *A $\{\delta(p^3+1), \delta; N, p^3\}$-minihyper, $N \geq 4$, $p = p_0^h$, p_0 prime, $p_0 \geq 7$, $p \geq 9$, $\delta \leq 2p^2 \Leftrightarrow 4p$, with total excess $e \leq p^3 \Leftrightarrow 4p$, is a sum of either:*
(1) *lines, (projected) $PG(3,p^{3/2})$'s (where the projection is from a point), and at most one (projected) $PG(5,p)$,*
(2) *lines, (projected) $PG(3,p^{3/2})$'s, and a $\{(p^2+p)(p^3+1), p^2+p; 3, p^3\}$-minihyper $\Omega \setminus N$, where Ω is a $PG(5,p)$ projected from a line L for which $\dim\langle L, L^p, L^{p^2}\rangle = 3$, and where N is the line contained in Ω.*

This result then was the building tool for the general classification result.

Theorem 6.2 (Ferret and Storme [4]) *Let F be a $\{\delta v_{\mu+1}, \delta v_\mu; N, p^3\}$-minihyper, $\mu \geq 2$, $\delta \leq 2p^2 \Leftrightarrow 4p$, $N \geq 3$, $p = p_0^h \geq 9$, $h \geq 1$, $p_0 \geq 7$ prime, with excess $e \leq p^2+p$.*

Then F is a sum of μ-dimensional spaces $PG(\mu, p^3)$, (projected) $PG(2\mu+1, \sqrt{q})$'s, and of at most one (projected) subgeometry $PG(3\mu+2, p)$.

References

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