Restrictions on Two-Weight Projective Linear Codes

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Abstract

To every two-weight linear code we associate a projective code. The original code is projective if and only if the associated code is a two-weight code. We then derive necessary conditions on those projective two-weight codes that are equivalent to their associated codes.

This extended abstract summarizes some recent work of the author concerning linear codes that have only two non-zero weights. To every two-weight code C is associated a projective code P (actually, a pair of projective codes). It is then shown that the original code C is itself projective if and only if the associated code P is a two-weight code. Finally, we derive necessary conditions on the parameters of C under the hypothesis that C is equivalent to P (or when C is equivalent to the complementary code of P).

1 Introductory material

Throughout this extended abstract, $\mathbb{F}_q$ will denote a finite field with q elements, $\mathbb{N}$ will denote the natural numbers $\{0, 1, \dots\}$, and $\mathbb{Q}$ the field of rational numbers. All codes will be linear over $\mathbb{F}_q$, and we will always work with Hamming weights.

Let M be a k-dimensional vector space over $\mathbb{F}_q$. Fix a basis for M, and let $\cdot$ denote the standard $\mathbb{F}_q$-valued dot product with respect to this basis. There is then an isomorphism between M and its linear dual space $M^\sharp = \mathrm{Hom}_{\mathbb{F}_q}(M, \mathbb{F}_q)$, where $x \in M$ maps to the linear functional $\lambda_x \in M^\sharp$ given by $\lambda_x(y) = x \cdot y$, $y \in M$.

We will describe linear codes using the notation of [13] and [14], as we now explain. Let $\mathcal{O}$ be the set of 1-dimensional linear subspaces of M, and let $\mathcal{O}^\sharp$ be the set of 1-dimensional linear subspaces of $M^\sharp$. Up to equivalence, a linear code with underlying vector space M is given by a pair $C = (M, \eta)$, where $\eta : \mathcal{O}^\sharp \to \mathbb{N}$ is a *multiplicity function*. The function η encodes the number of times a particular column will appear in a generator matrix for C. A linear code is *projective* if the only values of η are 0 or 1. Note that η having domain $\mathcal{O}^\sharp$ means that codes never have the zero functional as a coordinate functional (no zero columns in generator matrices).

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Remark 1.1 A number of authors have expressed linear codes in a manner similar to that above. I patterned the use of linear functionals after the treatment by Assmus and Mattson in [1]. I have discovered in retrospect that Slepian [9] and Peterson [7, pp. 41ff] (the second edition is [8]) use the phrase *modular representation* for what I call a multiplicity function. Peterson proves the $q = 2$ case of Proposition 1.2, as we will see below.

Multiplicity functions are also essentially equivalent to the *projective systems* of Tsfasman and Vlăduţ [10, p. 10] and the *projective multisets* of Dodunekov and Simonis [5, p. 1].

The projective codes associated to two-weight codes in Section 2 are special cases ($\sigma_\pm$) of the *dual transforms* discussed in [5, p. 14]. That paper's *σ-self-dual codes* are the subjects of the theorems in Section 3.

The notation $R[\mathcal{O}^\sharp]$ will mean the set of all functions from $\mathcal{O}^\sharp$ to R, and w denotes the Hamming weight defined on $\mathbb{F}_q$. (I.e., $w : \mathbb{F}_q \to \mathbb{N}$, with $w(0) = 0$ and $w(a) = 1$, for non-zero $a \in \mathbb{F}_q$.) There is a natural homomorphism

$$W : \mathbb{N}[\mathcal{O}^\sharp] \to \mathbb{N}[\mathcal{O}]$$

given by

$$W(\eta)(x) = \sum_{\lambda \in \mathcal{O}^\sharp} \eta(\lambda) w(\lambda(x)), \tag{1.1}$$

for $\eta \in \mathbb{N}[\mathcal{O}^\sharp]$, $x \in \mathcal{O}$. We will denote $W(\eta)$ by w_η for short. The function w_η is simply the Hamming weight function for the code $C = (M, \eta)$. Please be aware that it is possible for a non-zero x to have zero weight.

The homomorphism W extends to an isomorphism $W : \mathbb{Q}[\mathcal{O}^\sharp] \to \mathbb{Q}[\mathcal{O}]$, as described in [13] and [14]. We will show this directly by inverting the matrix representing W. As (1.1) makes clear, W is represented by a square matrix (which we will also call W) whose rows are indexed by elements $x \in \mathcal{O}$ and whose columns are indexed by linear functionals $\lambda \in \mathcal{O}^\sharp$. When we view the values of η as entries in a column vector with rows indexed by elements $\lambda \in \mathcal{O}^\sharp$, we then get the values of $W(\eta)$ to be entries in a column vector whose rows are indexed by elements $x \in \mathcal{O}$. Abusing notation, we can say that $W(\eta) = W\eta$, where the right hand side is matrix multiplication.

Making use of the isomorphism $M \to M^\sharp$, $x \mapsto \lambda_x$, described above, we will identify $\mathcal{O}^\sharp$ with $\mathcal{O}$; both are then identified with the set of one-dimensional subspaces of M. In terms of these identifications, the matrix W above is a square matrix of size $(q^k - 1)/(q - 1)$ with entries indexed by non-zero elements $x, y \in M$. (We will abuse notation by referring to an element of $\mathcal{O}$, i.e., a one-dimensional subspace of M, by one of the non-zero elements of M that it contains. This should not cause much confusion.) The entry in row x and column y is $w(\lambda_y(x)) = w(y \cdot x) = w(x \cdot y)$, for non-zero $x, y \in M$. We observe that this value is independent of the choices of representatives of elements of $\mathcal{O}$.

The following proposition is implicit in the work of Bogart, et al. [2] and is explicitly described over $\mathbb{F}_2$ by Peterson [7, p. 42].

Proposition 1.2 *Let M be a k-dimensional vector space defined over $\mathbb{F}_q$. Then the matrix W representing $W : \mathbb{Q}[\mathcal{O}^\sharp] \to \mathbb{Q}[\mathcal{O}]$ has inverse matrix W', whose entry at the position indexed by non-zero $x, y \in M$ is*

$$W'_{x,y} = \begin{cases} \frac{-(q-1)}{q^{k-1}}, & x \cdot y = 0, \\ \frac{1}{q^{k-1}}, & x \cdot y \neq 0. \end{cases}$$

Proof. Let $H_x = \ker \lambda_x$ be the hyperplane in M consisting of elements orthogonal to x. Observe that elements $x, y \in \mathcal{O}$ are equal, as elements of $\mathcal{O}$, if and only if $H_x = H_y$. Denote by $\bar{H}_x$ the set-theoretic difference $\bar{H}_x = M \setminus H_x$.

We first calculate the (x, x)-entry $P_{x,x}$ of the matrix product $P = WW'$. The sum $P_{x,x} = \sum_{z \in \mathcal{O}} W_{x,z} W'_{z,x}$ splits into two sums, one with z varying over (the one-dimensional subspaces of) H_x, the other with z varying over (the 'one-dimensional subspaces' of) $\bar{H}_x$. Since $W_{x,z} = 0$ for $z \in H_x$, the first sum vanishes. In the other sum, where $z \in \bar{H}_x$, we have $W_{x,z} = 1$ and $W'_{z,x} = 1/q^{k-1}$. Thus, $P_{x,x}$ equals $1/q^{k-1}$ times the number of 'one-dimensional subspaces' of $\bar{H}_x$. This latter number is the number of elements of $\bar{H}_x$ ($q^k - q^{k-1} = q^{k-1}(q-1)$) divided by $q - 1$; i.e., q^{k-1}. We conclude that $P_{x,x} = 1$.

Now suppose that $x \neq y$ as elements of $\mathcal{O}$. We decompose M into four subsets, each of which is closed under non-zero scalar multiplication. Table 1 below describes the subsets, the number of elements of $\mathcal{O}$ contained in each of the subsets, and the values of $W_{x,z}$ and $W'_{z,y}$ for z in the various subsets.

Table 1: Splitting M when $x \neq y$ in $\mathcal{O}$.

Subset	Number of elements in $\mathcal{O}$	$W_{x,z}$	$W'_{z,y}$
$H_x \cap H_y$	$(q^{k-2} - 1)/(q - 1)$	0	$-(q-1)/q^{k-1}$
$H_x \cap \bar{H}_y$	q^{k-2}	0	$1/q^{k-1}$
$\bar{H}_x \cap H_y$	q^{k-2}	1	$-(q-1)/q^{k-1}$
$\bar{H}_x \cap \bar{H}_y$	$q^{k-2}(q-1)$	1	$1/q^{k-1}$

The sum $P_{x,y} = \sum_{z \in \mathcal{O}} W_{x,z} W'_{z,y}$ then splits into four sums. Each sum is easy to evaluate, given the values in Table 1. The reader will then verify that $P_{x,y} = 0$ for $x \neq y$ in $\mathcal{O}$. $\square$

Remark 1.3 The fact that W has an inverse matrix W' is another way to prove that the homomorphism $W : \mathbb{Q}[\mathcal{O}^\sharp] \to \mathbb{Q}[\mathcal{O}]$ is actually an isomorphism. Moreover, the classical version $W : \mathbb{N}[\mathcal{O}^\sharp] \to \mathbb{N}[\mathcal{O}]$ is injective. This is essentially the proof of the MacWilliams extension theorem due to Bogart, et al. [2].

2 Projective codes associated to two-weight codes

Our objective is to use the isomorphism $W : \mathbb{Q}[\mathcal{O}^\sharp] \to \mathbb{Q}[\mathcal{O}]$ to study linear codes with only two non-zero weights. As above, we will identify $\mathcal{O}^\sharp$ and $\mathcal{O}$. Suppose $C = (M, \eta)$ is a linear code with exactly two non-zero weights; we assume that $\eta \in \mathbb{N}[\mathcal{O}]$, so that multiplicities and weights are natural numbers. Then there exist $a_1, a_2 \in \mathbb{N}$, $0 < a_1 < a_2$, such that $w_\eta(x) = a_1$ or $w_\eta(x) = a_2$ for $x \in \mathcal{O}$.

Let $S_i = \{x \in \mathcal{O} : w_\eta(x) = a_i\}$. By the two-weight assumption, $\mathcal{O}$ is the disjoint union of S_1 and S_2. Using the S_i we define two new linear codes. Let $P_i = (M, 1_{S_i})$, where 1_{S_i} is the indicator function for the subset $S_i \subset \mathcal{O}$. The codes P_i are projective codes. The codes P_1, P_2 are *complementary* projective codes in the sense their collections of coordinate functionals (as subsets of $\mathcal{O}^\sharp$) are disjoint and together give all possible coordinate functionals in $\mathcal{O}^\sharp$.

Let $\omega_1 = W(1_{S_1})$ be the weight function associated to the projective code P_1. The following theorem describes the multiplicity function η of C in terms of S_1, a_1, and a_2.

Theorem 2.1 *Let M be a k-dimensional vector space over $\mathbb{F}_q$. Let $C = (M, \eta)$ be a two-weight code, with weights $a_1 < a_2$ and associated subsets $S_i = \{x \in \mathcal{O} : w_\eta(x) = a_i\}$. Let s_1*

denote the number of elements of S_1. Let P_1 be the projective code $P_1 = (M, 1_{S_1})$, with weight function $\omega_1 = W(1_{S_1})$. Then the multiplicity function η on $\mathcal{O}$ (identified with $\mathcal{O}^\sharp$) satisfies

$$\eta(x) = \frac{a_1}{q^{k-1}}\left(q\omega_1(x) - (q-1)s_1\right) + \frac{a_2}{q^{k-1}}\left(1 + (q-1)s_1 - q\omega_1(x)\right). \tag{2.1}$$

Proof. Fix $x \in \mathcal{O}$. Among the s_1 elements $y \in S_1$, there are $\omega_1(x)$ of them satisfying $x \cdot y \neq 0$; the remaining $s_1 - \omega_1(x)$ elements $y \in S_1$ satisfy $x \cdot y = 0$. Since $H_x = \ker \lambda_x$ has $(q^{k-1}-1)/(q-1)$ one-dimensional subspaces, there are $(q^{k-1}-1)/(q-1) - s_1 + \omega_1(x)$ elements $y \in \mathcal{O} \setminus S_1$ that satisfy $x \cdot y = 0$. The remaining $(q^k-1)/(q-1) - (q^{k-1}-1)/(q-1) - \omega_1(x)$ elements $y \in \mathcal{O} \setminus S_1$ satisfy $x \cdot y \neq 0$. Aligning these counts with the appropriate values of entries in W' leads one to the following expression.

$$\begin{aligned}\eta(x) = & a_1\left[\frac{1}{q^{k-1}}\omega_1(x) - \frac{q-1}{q^{k-1}}(s_1 - \omega_1(x))\right] \\ & + a_2\left[\frac{1}{q^{k-1}}\left(\frac{q^k-1}{q-1} - \frac{q^{k-1}-1}{q-1} - \omega_1(x)\right)\right. \\ & \left. - \frac{q-1}{q^{k-1}}\left(\frac{q^{k-1}-1}{q-1} - s_1 + \omega_1(x)\right)\right].\end{aligned}$$

The reader will verify that this expression simplifies to the one in the statement of the theorem. □

Remark 2.2 When $a_1 = a_2$, the formula for η reduces to $\eta(x) = a_1/q^{k-1}$. This gives another proof of a theorem of Bonisoli [3] on linear one-weight codes. Other proofs of this result appear in [5, Proposition 4], [7, p. 43] (for $q = 2$), [11, Theorem 4], and [14, Remark 8.3].

Theorem 2.3 *If a two-weight code $C = (M, \eta)$ is projective, then its associated projective codes P_1 and P_2 are two-weight codes. Conversely, given a projective two-weight code P_1, there are unique values for $a_1 < a_2$ so that C is a projective two-weight code with associated code P_1.*

Proof. Since P_1 and P_2 are complementary projective codes, the weight functions ω_1 and ω_2 of P_1 and P_2 satisfy $\omega_1(x) + \omega_2(x) = q^{k-1}$, for all $x \in \mathcal{O}$. Thus, P_1 is a two-weight code if and only if P_2 is a two-weight code.

Suppose the two-weight code C is projective, i.e., $\eta(x) = 0$ or 1 for all $x \in \mathcal{O}$. It is clear from (2.1) that if $\omega_1(x) = \omega_1(y)$, then $\eta(x) = \eta(y)$. For the purposes of establishing a contradiction, suppose that P_1 is not a two-weight code. If ω_1 takes on only one value, then all the values of η are the same. That makes C a simplex code, which is a one-weight code.

For the other possibility, suppose ω_1 takes on values $b_1 < b_2 < \cdots < b_r$ on subsets $T_i = \{x \in \mathcal{O} : \omega_1(x) = b_i\}$. The set $\mathcal{O}$ is the disjoint union of the T_i. If $r > 2$, then η must take on the same value (0 or 1, say 1 for illustration) on at least two subsets, without loss of generality, say T_1 and T_2. Equation (2.1) then gives the following system of two equations in the variables a_1 and a_2 (the first equation holding for $x \in T_1$, the second for $x \in T_2$).

$$\begin{aligned}1 &= \frac{a_1}{q^{k-1}}\left(qb_1 - (q-1)s_1\right) + \frac{a_2}{q^{k-1}}\left(1 + (q-1)s_1 - qb_1\right) \\ 1 &= \frac{a_1}{q^{k-1}}\left(qb_2 - (q-1)s_1\right) + \frac{a_2}{q^{k-1}}\left(1 + (q-1)s_1 - qb_2\right)\end{aligned}$$

The system is invertible, because $b_1 < b_2$. Since the values of η on the left hand side are the same, we conclude that $a_1 = a_2$; contradiction.

For the converse, given that ω_1 takes on only values $b_1 < b_2$ on subsets T_1 and T_2, it is easy to solve the system of equations to find a_1 and a_2 that yield C as a projective two-weight code. In fact, one finds that it is necessary that $\eta(x) = 1$ for $x \in T_1$ and that $\eta(x) = 0$ for $x \in T_2$. Indeed, in the alternative, let us try to solve the following system of equations for a_1 and a_2. (The factor of q^{k-1} has been moved to the left side.)

$$\begin{aligned} 0 &= a_1\,(qb_1 - (q-1)s_1) + a_2\,(1 + (q-1)s_1 - qb_1) \\ q^{k-1} &= a_1\,(qb_2 - (q-1)s_1) + a_2\,(1 + (q-1)s_1 - qb_2) \end{aligned}$$

Subtracting the first equation from the second yields

$$q^{k-1} = a_1 q(b_2 - b_1) - a_2 q(b_2 - b_1) = -q(a_2 - a_1)(b_2 - b_1).$$

Since $a_1 < a_2$ and $b_1 < b_2$, the right side is negative, while the left side is positive. □

To summarize, suppose $C = (M, \eta)$ is a projective two-weight code, with weights $a_1 < a_2$ and $S_i = \{x \in \mathcal{O} : w_\eta(x) = a_i\}$. Then $P_1 = (M, 1_{S_1})$ is also a projective two-weight code, with weights $b_1 < b_2$ and $T_i = \{x \in \mathcal{O} : \omega_1(x) = b_i\}$. As the proof of Theorem 2.3 shows, $\eta = 1_{T_1}$, the indicator function for the subset $T_1 \subset \mathcal{O}$. Denote the numbers of elements in the sets S_i and T_i by s_i and t_i, respectively. Observe that the roles of C and P_1 are reversible. If we start with the projective two-weight code P_1, then its associated projective two-weight code is precisely C. This means that there is another set of equations like (2.1) with the roles of C and P_1 reversed. The consequences of these two sets of equations follow.

Theorem 2.4 *The following equations hold, relating the parameters of C and P_1.*

$$(a_2 - a_1)(b_2 - b_1) = q^{k-2} \tag{2.2}$$
$$(q-1)s_1 a_1 + (q^k - 1 - (q-1)s_1)a_2 = q^{k-1}(q-1)t_1 \tag{2.3}$$
$$(q-1)t_1 b_1 + (q^k - 1 - (q-1)t_1)b_2 = q^{k-1}(q-1)s_1 \tag{2.4}$$

Idea of proof. After a little manipulation, (2.1) becomes the following system of equations, the first one holding for $x \in T_1$, the second for $x \in T_2$.

$$q^{k-1} = a_1\,(qb_1 - (q-1)s_1) + a_2\,(1 + (q-1)s_1 - qb_1) \tag{2.5}$$
$$0 = a_1\,(qb_2 - (q-1)s_1) + a_2\,(1 + (q-1)s_1 - qb_2) \tag{2.6}$$

Subtracting these equations, as in proof of Theorem 2.3, yields (2.2).

The counterparts to (2.5) and (2.6), when the roles of C and P_1 have been reversed, are the following.

$$q^{k-1} = b_1\,(qa_1 - (q-1)t_1) + b_2\,(1 + (q-1)t_1 - qa_1) \tag{2.7}$$
$$0 = b_1\,(qa_2 - (q-1)t_1) + b_2\,(1 + (q-1)t_1 - qa_2) \tag{2.8}$$

Results (2.3) and (2.4) follow from manipulating and inverting (2.5)–(2.8). □

Equations (2.3) and (2.4) also follow from summing the weights of all the elements of C and P_1, respectively.

3 Restrictions on projective two-weight codes

We now state some restrictions on the parameters of a projective two-weight code C, under the hypothesis that C is equivalent to its associated projective two-weight code P_1 or to the complement P_2 of P_1. In fact, we need not assume that the codes are equivalent, just that they have the same parameters.

For technical reasons related to (2.2), we need to assume that q is not a square.

Theorem 3.1 *Assume q is not a square, and that a projective two-weight code C is equivalent to its associated projective two-weight code P_1. (More generally, just assume that $b_1 = a_1$, $b_2 = a_2$ and $t_1 = s_1$.) Then*

1. *k is even, say $k = 2t$;*
2. *$a_2 - a_1 = q^{t-1}$;*
3. *for some integer $u > 1$, $a_2 = q^{t-1}u$ and $s_1 = u(q^t - 1)/(q - 1)$.*

Idea of proof. From (2.2) and the hypotheses we have $(a_2 - a_1)^2 = q^{k-2}$. Since q is not a square, we conclude that k is even, say $k = 2t$, and that $a_2 - a_1 = q^{t-1}$. The remaining result follows from manipulating (2.2)–(2.4). □

Theorem 3.2 *Assume q is not a square, and that a projective two-weight code C is equivalent to its complementary associated projective two-weight code P_2. (More generally, just assume that $b_1 = q^{k-1} - a_2$, $b_2 = q^{k-1} - a_1$ and $t_1 = (q^k - 1)/(q - 1) - s_1$.) Then*

1. *k is even, say $k = 2t$;*
2. *$a_2 - a_1 = q^{t-1}$;*
3. *for some integer $u > 1$, $a_2 = q^{t-1}u$ and $(q - 1)t_1 = (q^t + 1)(u - 1)$.*

Idea of proof. The proof is similar to that of Theorem 3.1. □

Example 3.3 Suppose that M has dimension $k = 2t$ over $\mathbb{F}_2$, and that $E_1, E_2, \ldots, E_N$ are linear subspaces of dimension t. Suppose that $E_i \cap E_j = \{0\}$ for all $i \neq j$. Define a projective code $C = (M, \eta)$ by having η equal 1 on points of $\mathcal{O}$ lying on the union of the E_i, and η equaling 0 elsewhere. For $1 < N < 2^t + 1$, C is a projective two-weight code, with C having the same parameters as P_1, namely $a_1 = b_1 = (N - 1)2^{t-1}$, $a_2 = b_2 = N2^{t-1}$, and $s_1 = t_1 = N(2^t - 1)$. If $N = 1$, some nonzero elements have zero weight. If $N = 2^t + 1$, the union of the E_i equals all of $\mathcal{O}$, and C is a one-weight simplex code.

These codes arise in Dillon's partial spread family of bent functions [4].

Example 3.4 Let M have dimension $k = 4$ over $\mathbb{F}_2$. Let T_1 consist of five points: a basis for M (that's four points) and their sum. One then calculates that $t_1 = 5$, $a_1 = 2$, $a_2 = 4$, $s_1 = 10$, $b_1 = 4$, and $b_2 = 6$. The code C has the same parameters as the complementary code P_2.

These codes arise in connection with the Maiorana-McFarland bent functions [6], [12]. In fact, for dimension $k = 4$ over $\mathbb{F}_2$, there are 384 Maiorana-McFarland bent functions. Of these bent functions, 120 give rise to the code of Example 3.3 (with $N = 2$), 120 give rise to the complementary projective code to the code of Example 3.3, 72 give rise to the code of this example, and 72 give rise to the complementary projective code to the code of this example.

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