On identification of sets of vertices in the triangular grid

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Abstract

A subset C of vertices in a connected graph $G = (V, E)$ is called $(r, \leq l)$-identifying if for all subsets $L \subseteq V$ of size at most l, the sets $I(L)$, consisting of all the codewords which are within graphic distance r from at least one element in L, are different. The main result of the paper is that the minimum possible density of a $(1, \leq 2)$-identifying code in the triangular grid is $9/16$.

1 Introduction

Let $G = (V, E)$ be an undirected, connected graph, which may be finite or infinite, and C a nonempty subset of the vertex set V. For any vertex $v \in V$, and any $S \subseteq V$ we denote

$$B_r(v) = \{u \in V \mid d_G(u, v) \leq r\},$$

and

$$B_r(S) = \bigcup_{v \in S} B_r(v).$$

where $d_G(c, v)$ denotes the graphic distance between c and v, i.e., the number of edges on any shortest path between them. We further denote

$$I_r(v) = C \cap B_r(v),$$

and for every $L \subseteq V$,

$$I_r(L) = C \cap B_r(L).$$

We say that C is $(r, \leq l)$-identifying $(r > 0)$ if the sets $I_r(L)$ are different for all choices of L of size at most l. Because $I_r(\emptyset) = \emptyset$, all the sets $I_r(L)$ are nonempty, when L is nonempty. If V is finite, we define the *density* of C to be $|C|/|V|$. The elements of C are called *codewords*.

The study of identifying codes was initiated in [18], and such codes can be used in maintaining multiprocessor architectures using the following scheme. We can represent the multiprocessor architecture as a graph $G = (V, E)$: each vertex corresponds to a processor, and

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each edge to a bidirectional link between two vertices. Let us assume that at most l of the processors are malfunctioning. We choose a set of vertices — the code — and each vertex in the chosen set performs a test of its r-neighbourhood (the vertex itself included), and sends a central controller the message YES, if it detected problems, and NO, otherwise. So each vertex sends just one bit of information to the central controller. We would like to choose our set in such a way that based on the YES/NO answers from the elements of the chosen set, we can identify which (at most l) processors are malfunctioning.

A related problem, in which $r = 1$, and the chosen vertices do not test themselves, but only their neighbours, leads to the study of *locating-dominating sets*; see, e.g., [11] and [19].

In this paper we consider exclusively the triangular grid.

From now on distance always means the Euclidean distance unless stated otherwise.

The vertex set of the triangular grid T is

$$V = \{i(1,0) + j(\frac{1}{2}, \frac{\sqrt{3}}{2}) \mid i, j \in \mathbb{Z}\},$$

and there is an edge between any two points at distance one. We denote

$$v(i,j) = i(1,0) + j(\frac{1}{2}, \frac{\sqrt{3}}{2}).$$

The neighbours of $v(i,j)$ are therefore the points

$$v(i-1, j+1), v(i, j+1), v(i+1, j), v(i+1, j-1), v(i, j-1), v(i-1, j). \tag{1}$$

Denote by R_n the set of vertices $v(i,j)$ with $|i| \leq n$ and $|j| \leq n$. The density of a code C in T is defined to be

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap R_n|}{|R_n|}.$$

We also consider the graphs $T_n = (V(T_n), E(T_n))$, $n \geq 3$: their vertex set $V(T_n)$ consists of all $v(i,j)$ with $i, j \in \{0, 1, \ldots, n-1\}$ and as in T, every vertex $v(i,j)$ has the six neighbours in (1), but now the indices are modulo n. We say that T_n is obtained from the subgraph of T with vertex set $\{v(i,j) \mid i, j = 0, 1, \ldots, n-1\}$ by wrapping around.

In general, it is known [1] that the density of an $(r, \leq 1)$-identifying code is at least $2/(6r+3)$, and that there exists an $(r, \leq 1)$-identifying code with density $1/(2r+4)$ if $r \equiv 0 \mod 4$, and density $1/(2r+2)$ otherwise. For small values, better bounds can be found in [3]. The best possible density of a $(1, \leq 1)$-identifying code is $1/4$; see [18].

In this paper we show that the optimal density of a $(1, \leq 2)$-identifying code in the infinite triangular grid T is $9/16$. For $l \geq 3$, no $(r, \leq l)$-identifying codes exist (cf. Section 2). We also show that if $n \geq 16$ is divisible by four, then the smallest $(1, \leq 2)$-identifying code in the graphs T_n has $9n^2/16$ codewords. The results of this paper are from [14].

Several other architectures have also been considered: in particular the square lattice (see [18], [1], [3], [6], [7], [10], [12] and [15]), the king lattice (see [2], [9], [13]) and the hexagonal mesh (see [9], [3], [8]). For identifying codes in binary hypercubes, see, e.g., [18], [16] and their references. Complexity issues have been considered in [4], [5], [10], [16] and [17].

2 The results

Given two vertices u and v, we say that u covers v, and vice versa, if u and v are neighbours or they are the same vertex.

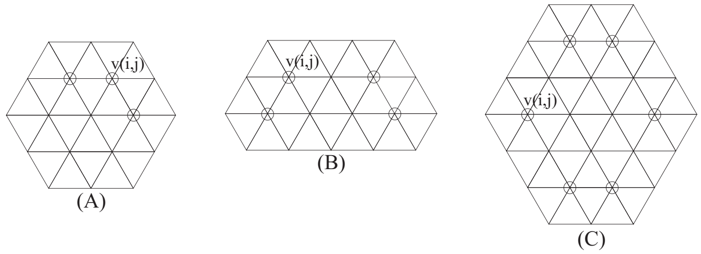

Figure 1: The three patterns.

Because always

$$I_r(v(i,j), v(i,j-2)) = I_r(v(i,j), v(i,j-1), v(i,j-2)),$$

there are no $(r, \leq l)$-identifying codes for $l \geq 3$.

From now on we consider the case $l = 2$ and $r = 1$. We write $I(v)$ and $I(L)$ instead of $I_1(v)$ and $I_1(L)$; and refer to them as the I-sets of v and L (with respect to C).

The next theorem states that if a code is $(1, \leq 2)$-identifying, then there is at least one codeword in each of the patterns depicted in Figure 1 (and the ones obtained from them by rotations) and vice versa.

Let ρ_k ($k \in \mathbb{Z}$) denote a rotation of the graph T by an angle of $\pi k/3$ (counter-clockwise) with respect to the origin. The image of a set $S \subseteq V$ under the mapping ρ_k is denoted by $\rho_k(S)$. Notice that we consider the graph drawn as in Figure 1 and hence always $\rho_k(S) \subseteq V$.

Theorem 2.1. *[14] A code $C \subseteq V$ is $(1, \leq 2)$-identifying if and only if it has properties (A), (B) and (C), by which we mean that for every $(i,j) \in \mathbb{Z}^2$ and $k \in \{0,1,2,3,4,5\}$, the sets*

$$\rho_k(v(i,j), v(i-1,j), v(i+1,j-1)) \quad (A)$$
$$\rho_k(v(i,j), v(i,j-1), v(i+2,j), v(i+3,j-1)), \quad (B)$$
$$\rho_k(v(i,j), v(i,j+2), v(i+1,j+2), v(i+3,j), v(i+3,j-2), v(i+2,j-2)) \quad (C)$$

each contain at least one codeword of C.

It is easy to check that the code of Figure 2 has the properties (A), (B) and (C), and therefore we obtain the following result.

Theorem 2.2. *The code of Figure 2 is $(1, \leq 2)$-identifying and has density $9/16$.*

The fact that this construction is optimal comes from Theorem 2.5.

From now we assume that $n \geq 11$ for T_n. A result similar to Theorem 2.1 is valid for the graphs T_n.

Definition 2.1. *Assume that C is a code in T_n. Define*

$$C^\infty = \{v(i,j) \in T \mid v(i \bmod n, j \bmod n) \in C\},$$

where $k \bmod n$ denotes the least non-negative residue of k modulo n.

We say that code C has property (A) (resp. (B), (C)) if C^∞ does.

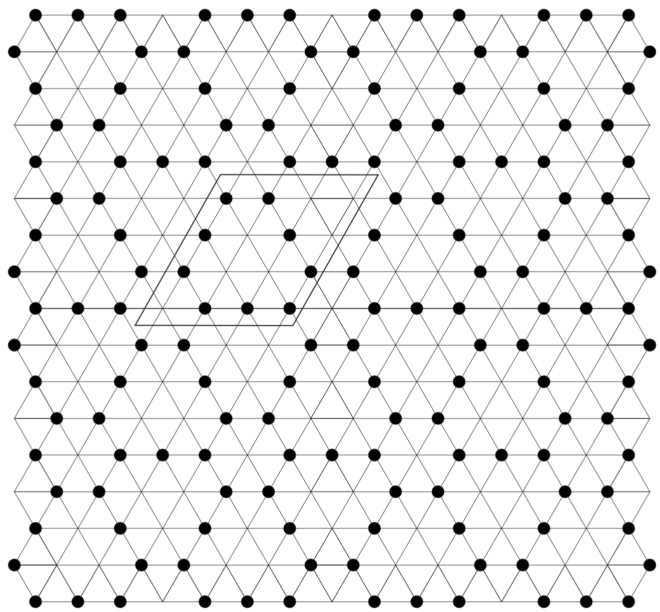

Figure 2: An optimal $(1, \leq 2)$-identifying code (part). Codewords are marked by black circles.

Theorem 2.3. *[14] A code $C \subseteq T_n$, $n \geq 11$, is $(1, \leq 2)$-identifying in T_n if and only if it has the properties (A), (B) and (C).*

Assume that $C \subseteq T_n$. Let N denote the set of non-codewords among the vertices.

If the condition (A) is satisfied by $C \subseteq V(T_n)$, then

$$|I(x)| \geq \begin{cases} 4 & \text{if } x \in N \\ 3 & \text{if } x \in C \end{cases}$$

and this implies that

$$7|C| \geq 3|C| + 4|N| = 3|C| + 4(n^2 - |C|),$$

and hence C must have density at least $1/2$. For n even, one can construct such a code by taking all the points on every second horizontal row as codewords. Using the method introduced later, the lower bound $1/2$ also works for the infinite graph T. The pattern (B) does not change the situation. However, the condition (C) alters the game a lot.

Theorem 2.4. *For $n \geq n_0$, the density of a $(1, \leq 2)$-identifying code in T_n is at least $9/16$.*

Proof. The rather technical proof is omitted here (for the complete proof see [14] pages 11–21). □

Using this we now easily obtain the main result of this paper.

Theorem 2.5. *The smallest possible density of a $(1, \leq 2)$-identifying code in the infinite triangular grid T is $9/16$.*

Proof. Let C be a $(1, \leq 2)$-identifying code in T. Assume that $n \geq n_0$, where n_0 is as in Theorem 2.4. We claim that the code

$$F = (R_{n+2} \setminus R_n) \cup (C \cap R_n)$$

viewed as a code in T_{2n+5} is $(1, \leq 2)$-identifying. The code F consists of the "frame" $R_{n+2} \setminus R_n$ and the "picture" $C \cap R_n$. In F^∞ we paste together infinitely many copies of F. If two vertices of T belong to different copies of the picture, they are at least graphic distance five apart (because the frame consists of two full layers around the picture, and one has to cross two frames). Consequently, no pattern (A), (B) or (C) in T can contain points from two different pictures of F^∞. Hence, every pattern (A), (B) and (C) in T hits one of the frames (and then certainly contains a codeword of F^∞) or lies completely within one of the pictures (and again contains at least one codeword, because C has the properties (A), (B) and (C)). This shows that F is $(1, \leq 2)$-identifying in T_{2n+5} by Theorem 2.3, and by Theorem 2.4,

$$|C \cap R_n| \geq |F| - 8(2n+3) \geq \frac{9}{16}(2n+5)^2 - 8(2n+3)$$

and hence

$$\limsup_{n \to \infty} \frac{|C \cap R_n|}{|R_n|} \geq \frac{9}{16}.$$

□

It is also interesting to consider the graphs T_n themselves and prove a finite version of Theorem 2.5.

Theorem 2.6. *If $n \geq 16$ and n is divisible by 4, then the minimum possible cardinality of a $(1, \leq 2)$-identifying code in T_n equals $9n^2/16$.*

Proof. If $C \subseteq T_n$ is $(1, \leq 2)$-identifying, then by Theorem 2.3 it has the properties (A), (B) and (C), i.e., C^∞ satisfies (A), (B) and (C). By Theorem 2.1, C^∞ is $(1, \leq 2)$-identifying. The density of C is the same as the density of C^∞, and hence at least $9/16$ by Theorem 2.5.

To obtain a $(1, \leq 2)$-identifying code $C \subseteq T_n$ with cardinality $9n^2/16$ we can take the intersection of Figure 2 with T_n: the periodicity properties of Figure 2 guarantee that there are no problems with the wrapping around. □

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