Results on linear codes meeting the Griesmer bound from results on t-fold $(N-K)$-blocking sets in $PG(N,q)$

S. Ferret*, L. Storme, P. Sziklai and Zs. Weiner

Abstract

Minihypers in finite projective spaces have been used greatly to obtain results on linear codes meeting the Griesmer bound. Minihypers are particular t-fold $(N-K)$-blocking sets in finite projective spaces $PG(N,q)$. Our goal is to use characterization results on t-fold $(N-K)$-blocking sets to obtain new characterization results on minihypers; thus leading to new results on linear codes meeting the Griesmer bound.

1 Introduction

In coding theory, the Griesmer bound [8, 16] states that if there exists a linear $[n,k,d;q]$ code for given values of k,d and q, then $n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil = g_q(k,d)$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

The question arises whether there exists a linear $[n,k,d;q]$ code whose length n is equal to the lower bound $g_q(k,d)$. This coding-theoretical problem can be translated into a problem on *minihypers in projective spaces*. Let $PG(N,q)$ be the N-dimensional projective space over the finite field of order q.

Definition 1 (Hamada and Tamari [13]) *Let F be a set of f points in $PG(N,q)$, where $N \geq 2$ and $f \geq 1$. If $|F \cap H| \geq m$ for every hyperplane H in $PG(N,q)$ and $|F \cap H| = m$ for some hyperplane of $PG(N,q)$, then F is called an $\{f,m;N,q\}$-minihyper.*

Hamada showed that for $d = q^{k-1} - \sum_{i=1}^{h} q^{\lambda_i}$, there is a one-to-one correspondence between the set of all non-equivalent $[n,k,d;q]$ codes meeting the Griesmer bound and the set of all projectively distinct $\{\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; k-1, q\}$-minihypers [9]; where $v_l = (q^l-1)/(q-1)$, for any integer $l \geq 0$. Let $G = (g_1 \cdots g_n)$ be a generator matrix for a linear $[n,k,d;q]$ code, $d < q^{k-1}$, meeting the Griesmer bound. Then the set $PG(k-1,q) \setminus \{g_1, \ldots, g_n\}$ is the minihyper linked to the linear code meeting the Griesmer bound. This idea of studying the minihypers linked to linear codes meeting the Griesmer bound is in fact the idea of using *anticodes* ([15, Ch. 17,§6] and [4]).

The classical examples of linear codes meeting the Griesmer bound are of Belov, Logachev and Sandimirov. We describe them by using the link with minihypers. Consider in $PG(k-1,q)$ a disjoint union of ϵ_0 different points, ϵ_1 disjoint lines, ..., ϵ_{k-2} disjoint $(k-2)$-dimensional subspaces, where $0 \leq \epsilon_i \leq q-1$ for $0 \leq i \leq k-2$. Then such a set

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defines a $\{\sum_{i=0}^{k-2}\epsilon_i v_{i+1}, \sum_{i=0}^{k-2}\epsilon_i v_i; k-1, q\}$-minihyper. The linear codes associated to these minihypers are the linear codes meeting the Griesmer bound, discovered by Belov, Logachev and Sandimirov [1].

Strong classification results on minihypers, so on linear codes meeting the Griesmer bound, for general values of n, k, d and q, were obtained by Hamada, Helleseth and Maekawa, and by Ferret and Storme.

Theorem 2 (a) (Hamada, Helleseth, Maekawa [11, 12]) *A* $\{\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; k-1, q\}$-*minihyper, with* $h \le \sqrt{q}$, *is the pairwise disjoint union of a* λ_0-*,...,*λ_h*-dimensional subspace of* $PG(k-1,q)$.

(b) (Ferret and Storme [5]) *Let* F *be a* $\{\sum_{i=0}^{s}\epsilon_i v_{i+1}, \sum_{i=0}^{s}\epsilon_i v_i; k-1, q\}$*-minihyper, where* $\sum_{i=0}^{s}\epsilon_i < 2\sqrt{q}$, $q > q_0$. *Then* F *consists of the pairwise disjoint union of either:*
(1) ϵ_s *subspaces* $PG(s,q), \epsilon_{s-1}$ *subspaces* $PG(s-1,q), \ldots, \epsilon_0$ *points,*
(2) *one subgeometry* $PG(2l+1,\sqrt{q})$, *for some integer* l *with* $1 \le l \le s$, ϵ_s *subspaces* $PG(s,q), \ldots, \epsilon_{l+1}$ *subspaces* $PG(l+1,q), \epsilon_l - \sqrt{q} - 1$ *subspaces* $PG(l,q), \epsilon_{l-1}$ *subspaces* $PG(l-1,q), \ldots, \epsilon_0$ *points,*
(3) *one subgeometry* $PG(2l,\sqrt{q})$, *for some integer* l *with* $1 \le l \le s$, ϵ_s *subspaces* $PG(s,q), \ldots,$ ϵ_{l+1} *subspaces* $PG(l+1,q), \epsilon_l - 1$ *subspaces* $PG(l,q), \epsilon_{l-1} - \sqrt{q}$ *subspaces* $PG(l-1,q), \epsilon_{l-2}$ *subspaces* $PG(l-2,q), \ldots, \epsilon_0$ *points.*

So the results of Hamada, Helleseth and Maekawa show that the linear codes meeting the Griesmer bound, corresponding to the minihypers satisfying the conditions of Theorem 2 (a), are of Belov-Logachev-Sandimirov type.

For $\sum_{i=0}^{s}\epsilon_i \ge \sqrt{q}+1$, q square, new examples of minihypers, so of linear codes meeting the Griesmer bound, start appearing (see Theorem 2 (b)).

In Theorem 2 (b), the main difficulty in improving the results of Hamada, Helleseth and Maekawa is the possible occurrence of Baer subgeometries $\Pi_r = PG(r,\sqrt{q})$ in the minihyper when q is square. When such a Baer subgeometry $PG(r,\sqrt{q})$ occurs within the minihyper, it is reconstructed from the planes $PG(2,\sqrt{q})$ inside Π_r. The methods of [5] however only make this possible for $\sum_{i=0}^{s}\epsilon_i < 2\sqrt{q}$.

Recently, a new approach for studying minihypers has been used. This new approach involves the use of a closely related geometrical structure; namely that of a *t-fold $(N-K)$-blocking set in $PG(N,q)$*.

2 Multiple blocking sets in finite projective spaces

Definition 3 *A* t-fold $(N-K)$-blocking set *in* $PG(N,q)$ *is a set* B *of points of* $PG(N,q)$ *intersecting every* K*-dimensional subspace in at least* t *points. A* t*-fold* $(N-K)$*-blocking set* B *of* $PG(N,q)$ *is called* minimal *when no proper subset of* B *is still a* t*-fold* $(N-K)$*-blocking set.*

A 1*-fold* $(N-K)$*-blocking set of* $PG(N,q)$ *is also simply called an* $(N-K)$-blocking set *of* $PG(N,q)$.

Classical examples of $(N-K)$-blocking sets of $PG(N,q)$ are:

1. A subspace $PG(N-K,q)$ of $PG(N,q)$.

2. Cones with an m-dimensional vertex π_m and base a Baer subspace $PG(2(N-K-m-1),\sqrt{q})$, for some m with $\max\{-1, N-2K-1\} \leq m \leq N-K-1$. These cones are called *Baer-cones* and will be denoted by $\langle \pi_m, PG(2(N-K-m-1),\sqrt{q})\rangle$.

The Bose-Burton theorem states that the subspaces $PG(N-K,q)$ of $PG(N,q)$ are the smallest $(N-K)$-blocking sets of $PG(N,q)$.

Theorem 4 (Bose and Burton [3]) *The subspaces $PG(N-K,q)$ are the smallest $(N-K)$-blocking sets of $PG(N,q)$.*

Recently, Bokler managed to include Baer-cones in a general characterization result on $(N-K)$-blocking sets.

Theorem 5 (Bokler [2]) *Suppose B is a minimal $(N-K)$-blocking set of $PG(N,q)$, $N \geq 2$, q square and $q \geq 16$. If $|B| \leq v_{N-K+1} + \sqrt{q}v_{N-K}$, then B contains a Baer-cone with an m-dimensional vertex and base a Baer subspace $PG(2(N-K-m-1),\sqrt{q})$, for some m with $\max\{-1, N-2K-1\} \leq m \leq N-K-1$.*

There is an obvious way of constructing t-fold $(N-K)$-blocking sets in $PG(N,q)$; simply consider a union of t pairwise disjoint different $(N-K)$-blocking sets in $PG(N,q)$. For instance, a disjoint union of t of the classical examples of $(N-K)$-blocking sets in $PG(N,q)$ considered above.

The examples obtained in this way also have the property of having a *small cardinality*.

The question arises whether it is possible to characterize, for t *small*, minimal t-fold $(N-K)$-blocking sets in $PG(N,q)$ of *small* cardinality as being the disjoint union of t of the classical examples of $(N-K)$-blocking sets in $PG(N,q)$ considered above.

This is indeed possible.

The crucial part of our techniques was a generalization of the following important result on minimal $(N-K)$-blocking sets.

Theorem 6 (Szőnyi and Weiner [17]) *Let B be a minimal $(N-K)$-blocking set in $PG(N,q)$, $q=p^h$, $p>2$ prime, $h \geq 1$, of size less than $3(q^{N-K}+1)/2$. Then every subspace that intersects B in at least one point, intersects B in $1 \pmod p$ points.*

This $1 \pmod p$ result gives important information which can be used to obtain characterization results on such minimal $(N-K)$-blocking sets.

3 New results on t-fold $(N-K)$-blocking sets

The results of the preceding theorem have been extended to the following theorem.

Theorem 7 (Ferret, Storme, Sziklai and Weiner [7]) *Let B be a minimal t-fold $(N-K)$-blocking set in $PG(N,q)$, $q=p^h$, $p>2$ prime, $h\geq 1$, of size less than $(t+3/2)(q^{N-K}+1)$. Then every K-dimensional subspace intersects B in $t \pmod p$ points, and any subspace of dimension less than K intersects B in $0,1,\ldots,t \pmod p$ points.*

The preceding result plays the crucial role in our techniques. For t *small*, let B be a minimal t-fold $(N-K)$-blocking set in $PG(N,q)$ of *small* cardinality. The preceding result enables us to obtain a lot of information on the possible intersections of B with the subspaces of $PG(N,q)$. And it enables us to reconstruct all the t different minimal $(N-K)$-blocking sets in $PG(N,q)$ which form the t-fold $(N-K)$-blocking set B. We obtained the following characterization result on minimal t-fold $(N-K)$-blocking sets.

Theorem 8 (Ferret, Storme, Sziklai and Weiner [7]) *Let B be a minimal t-fold $(N-K)$-blocking set in $PG(N,q)$, $q=p^h$, p prime, $h \geq 1$, where $t \leq p/2$, and where $|B| \leq t|PG(2(N-K),\sqrt{q})|$.*

Then B is the union of t pairwise disjoint subspaces $PG(N-K,q)$ and/or Baer-cones.

The goal of theorems of this type on t-fold $(N-K)$-blocking sets in $PG(N,q)$ is to obtain improvements to the known results (Theorem 2) on minihypers in finite projective spaces.

4 Results on minihypers following from results on t-fold $(N-K)$-blocking sets

Minihypers in finite projective spaces are particular examples of t-fold $(N-K)$-blocking sets in projective spaces.

Theorem 9 (Hamada [10, Theorem 2.5]) *Let K be any integer, $1 \leq K < n$. If F is a $\{\sum_{i=0}^{N-K}\epsilon_i v_{i+1}, \sum_{i=1}^{N-K}\epsilon_i v_i; N, q\}$-minihyper, with $0 \leq \epsilon_i \leq q-1$, $i=0,\ldots,N-K$, with $\epsilon_{N-K} \neq 0$, then F is an ϵ_{N-K}-fold $(N-K)$-blocking set in $PG(N,q)$.*

This means that results on (multiple) $(N-K)$-blocking sets are of interest for linear codes meeting the Griesmer bound.

For instance, as indicated in the title of the article of Bose and Burton [3], the characterization result of Theorem 4 proves the uniqueness of the MacDonald codes. More precisely, the result of Theorem 4 characterizes the $\{v_{N-K+1}, v_{N-K}; N, q\}$-minihypers as being subspaces $PG(N-K,q)$. Since the minihypers are completely characterized, the corresponding linear codes meeting the Griesmer bound are completely characterized, and these are the MacDonald codes.

We concentrate on $\{tv_{N-K+1} + \sum_{i=0}^{N-K-1}\epsilon_i v_{i+1}, tv_{N-K} + \sum_{i=0}^{N-K-1}\epsilon_i v_i; N, q\}$-minihypers F, with $|F| \leq t|PG(2(N-K),\sqrt{q})|$, with $t \leq p/2$ and with $\sum_{i=0}^{N-K-1}\epsilon_i \leq t\sqrt{q}$.

These minihypers are t-fold $(N-K)$-blocking sets in $PG(N,q)$, but a priori, it is not known whether they are *minimal* t-fold $(N-K)$-blocking sets or *non-minimal* t-fold $(N-K)$-blocking sets.

Namely, for certain parameters, minihypers can be non-minimal (multiple) $(N-K)$-blocking sets.

For instance, consider a minihyper F in $PG(4,q)$ which is the disjoint union of a plane α and a line L. Then F is a $\{q^2+2q+2, q+2; 4, q\}$-minihyper. And F is a 2-blocking set of $PG(4,q)$ since it intersects every plane, but it is non-minimal as a 2-blocking set since the occurrence of the plane α is sufficient for F to be a 2-blocking set.

One must pay attention to this fact that certain minihypers could be non-minimal (multiple) $(N-K)$-blocking sets, for the result of Theorem 8 is formulated for **minimal** (multiple) $(N-K)$-blocking sets.

We therefore proceed as follows to obtain results on minihypers using the results on minimal (multiple) $(N-K)$-blocking sets in $PG(N,q)$.

Theorem 9 states that a $\{tv_{N-K+1}+\sum_{i=0}^{N-K-1}\epsilon_i v_{i+1}, tv_{N-k}+\sum_{i=1}^{N-K-1}\epsilon_i v_i; N, q\}$-minihyper F, with $0 \leq \epsilon_i \leq q-1$, $i=0,\ldots,N-K-1$, $t \neq 0$, is a t-fold $(N-K)$-blocking set in $PG(N,q)$.

Then F might be a minimal or non-minimal t-fold $(N-K)$-blocking set in $PG(N,q)$, but in either case, it contains a minimal t-fold $(N-K)$-blocking set in $PG(N,q)$. We selected the parameters in the description of F in such a way that Theorem 8 can be applied.

This then gives us a limited number of possibilities for a large subset of F; described as a disjoint union of t subspaces $PG(N-K,q)$ and/or Baer-cones $\langle \pi_{m_i}, PG(2(N-K-m_i-1),\sqrt{q})\rangle$, $-1 \leq m_i \leq N-K-2$. We investigated the different possibilities and we were able to eliminate the Baer-cones $\langle \pi_{m_i}, PG(2(N-K-m_i-1),\sqrt{q})\rangle$ for which $0 \leq m_i \leq N-K-2$. This is formulated in the following theorem.

Theorem 10 (Ferret, Storme, Sziklai and Weiner [7]) *A $\{tv_{N-K+1}+\sum_{i=0}^{N-K-1}\epsilon_i v_{i+1}, tv_{N-K}+\sum_{i=0}^{N-K-1}\epsilon_i v_i; N, q\}$-minihyper F, with $|F| \leq t|PG(2(N-K),\sqrt{q})|$, with $t \leq p/2$ and with $\sum_{i=0}^{N-K-1}\epsilon_i \leq t\sqrt{q}$, contains a minimal t-fold $(N-K)$-blocking set of $PG(N,q)$ which is the union of t pairwise disjoint subspaces $PG(N-K,q)$ and/or subgeometries $PG(2(N-K),\sqrt{q})$.*

This sometimes makes it possible to completely characterize the corresponding minihyper. We explain this by giving a particular characterization result.

Theorem 11 (Ferret, Storme, Sziklai and Weiner [7]) *A $\{t(q^2+q+1)+\epsilon_1(q+1)+\epsilon_0, t(q+1)+\epsilon_1; 4, q\}$-minihyper F, with $t \geq 2$, t small, and $\epsilon_1+\epsilon_0 \leq t\sqrt{q}$, is the union of t pairwise disjoint $PG(4,\sqrt{q})$.*

Proof: Using Theorem 10, it is known that F contains the union of t pairwise disjoint planes $PG(2,q)$ and/or subgeometries $PG(4,\sqrt{q})$.

It is impossible that there is a plane in this disjoint union since two planes in $PG(4,q)$ always intersect in at least a point, and a plane always intersects a subgeometry $PG(4,\sqrt{q})$.

So, F contains the union of t pairwise disjoint subgeometries $PG(4,\sqrt{q})$.

Then, from the condition $\epsilon_1+\epsilon_0 \leq t\sqrt{q}$, it follows $\epsilon_1 = t\sqrt{q}$ and $\epsilon_0 = 0$. Hence, F is equal to the union of t pairwise disjoint Baer subgeometries $PG(4,\sqrt{q})$. $\square$

This latter result already gives a new characterization result on minihypers, so on linear codes meeting the Griesmer bound (see also Theorem 2).

For minihypers with more general parameters, the research is still going on. We expect to obtain complete characterizations for $\{tv_{N-K+1}+\sum_{i=0}^{N-K-1}\epsilon_i v_{i+1}, tv_{N-K}+\sum_{i=0}^{N-K-1}\epsilon_i v_i; N, q\}$-minihyper F, with $|F| \leq t|PG(2(N-K),\sqrt{q})|$, with $t \leq p/2$ and with $\sum_{i=0}^{N-K-1}\epsilon_i \leq t\sqrt{q}$, for which the parameters ϵ_i, $i=0,\ldots,N-K-1$, satisfy certain supplementary conditions.

5 Links with results of Ward

Recently, the following result on the weights of linear codes over prime fields meeting the Griesmer bound was proven by Ward [18].

Theorem 12 (Ward [18]) *Let C be a linear $[n,k,d;p]$ code over the prime field $GF(p)$ meeting the Griesmer bound. If p^e is the maximal power of the characteristic p which divides the minimum distance d of C, then all the weights of C are divisible by p^e.*

This result imposes severe conditions on the possible weights of C when p^e divides the minimum distance d.

The importance of Theorems 6 and 7 is similar to the importance of the conditions on the possible weights of the linear code meeting the Griesmer bound, described in the preceding theorem.

Every minihyper which is also a minimal (multiple) $(N-K)$-blocking set in $PG(N,q)$ satisfying the conditions of Theorems 6 and 7 has by these theorems conditions on the possible intersection sizes with the subspaces.

This information plays a crucial role in characterizing these minihypers.

Since we are studying linear codes meeting the Griesmer bound using a geometrical approach, we also wish to remark that an alternative proof of the above mentioned theorem of Ward on the divisibility of linear codes meeting the Griesmer bound was presented by Landjev and Rousseva [14] at the *Third EuroWorkShop on Optimal Codes and Related Topics*, June 10-16, 2001, Sunny Beach, Bulgaria. This alternative proof also uses a geometrical approach to linear codes.

6 Remark

We wish to remark that the results of this abstract were also presented at the *Eight International Workshop Algebraic and Combinatorial Coding Theory*, September 8-14, 2002, Tsarskoe Selo, Russia [6].

At that conference, the general ideas of the methods were presented. The new results presented in this extended abstract are the determination of the upper bounds on t and $|B|$ in the statement of Theorem 8, and the exact description of the minimal t-fold blocking sets inside the minihypers described in Theorem 10.

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Address of the authors:

S. Ferret and L. Storme: Ghent University, Dept. of Pure Maths and Computer Algebra, Krijgslaan 281, 9000 Ghent, Belgium. (saferret@cage.rug.ac.be, http://cage.rug.ac.be/∼saferret) (ls@cage.rug.ac.be, http://cage.rug.ac.be/∼ls)

P. Sziklai and Zs. Weiner: Eötvös Loránd University, Dept. of Computer Science, Pázmány P. sétány 1/c, Budapest, Hungary H-1117 (sziklai@cs.elte.hu) (weiner@cs.elte.hu)