Tilings of closed surfaces by Steiner triple systems *

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Abstract

A Steiner triple system of order n (briefly $STS(n)$) is a family of 3-element subsets (blocks) of a set $N = \{1, 2, \ldots, n\}$ such that each 2-element subset of N appears in exactly one block. We will identify every block $(i, j, k) \in STS(n)$ with a topological triangle with vertices i, j and k. Glueing together all equal edges of all triangles corresponding to all triples of two nonintersecting Steiner triple systems one can get a closed surface or closed pseudo-surface. A class of tilings of a nonorientable closed surface (a sphere with $(n-3)(n-4)/6$ crosscaps) by special types of pairs of Steiner triple systems of order $n \equiv 3 \pmod 6$ is presented. It is also shown that there exists a class of tilings of a nonorientable closed surface by STS's of order n for half of the residue class $n \equiv 1 \pmod 6$.

1 Introduction

The paper is devoted to tilings of closed surfaces by pairs of Steiner triple systems. A class of tilings of a nonorientable closed surface (a sphere with $(n-3)(n-4)/6$ crosscaps) by special types of pairs of Steiner triple systems of order $n \equiv 3 \pmod 6$ is presented. [1] It is also shown that there exists a class of tilings of a nonorientable closed surface by STS's of order n for half of the residue class $n \equiv 1 \pmod 6$. Such tilings are also interpreted in the literature as triangular embeddings of complete graphs in closed surfaces. We consider that the notion of a tiling is more suitable (it will be clear from definitions given below). The investigation of these tilings has stirring and long history since 1852, see [2]. The search gave new directions concerning embeddings (or tilings) in many subjects such as design theory, graph theory, combinatorial topology, algebra, classical geometry, see all links in [3] and surveys in [2, 3, 4, 6]. Despite the existence of many results devoted to embeddings of a complete graph, K_n, in a

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[1] It should be mentioned some parallelism between the results given here and the main result which was published by the author in [1], where special components (called i-components) of perfect binary codes are investigated. In the paper [1] the construction of a perfect code (of length $n = 2^m - 1$) partitioning into two indecomposable i-components is presented. In both situations (for closed surfaces and for perfect codes) we have a pair of structures such that their union gives a perfect structure (in the case of closed surface without punched points we also have in some sense a "perfectness"). In both cases it is necessary to guarantee a connectivity (in one case for surfaces and in the other for i-components). As a basis of each construction it was used a pair of special type nonintersected Steiner triple system (of course these pairs of STS's are extremely different).

closed surface there still remain many unsolved problems, for example it is not known if there exists nonisomorphic face 2-colorable triangulation of a nonorientable surface by a complete graph K_n for every $n \equiv 1 \pmod 6$. It is also interesting to enumerate all nonisomorphic face 2-colorable triangulations of an orientable or a nonorientable surface by a complete graph K_n for every admissible n.

It is well known [2] that a complete graph K_n triangulates some orientable surface if and only if $n \equiv 0, 3, 4$ or $7 \pmod{12}$ and triangulates some nonorientable surface if and only if $n \equiv 0$ or $1 \pmod 3$, $n > 7$. A triangulation is *face 2-colorable* if the triangular faces of the embedding can be *properly 2-coloured* in black and white colours such that no two monochromatic triangles share an edge. For the embeddings to be face 2-colorable it is necessary for n to be odd because the vertex degrees should be even. The case of 2-coloroubility is of special interest for design theory because all monochromatic triangles on the surface induce a Steiner triple system (sets of three vertices giving the triangles in each of these two colour sets form two Steiner triple systems). So we immediately get a tiling of the surface given by black and white Steiner triple systems of order n. Further we will call such a pair of Steiner triple systems of order n a *tiling of order* n (it is also called bi-embedded). We remind that a Steiner triple system of order n (briefly $STS(n)$) is a family of 3-element blocks (subsets or triples) of the set $N = \{1, 2, \ldots, n\}$ such that each inordered pair of elements of N appears in exactly one block. Two STS's of order n are called *equivalent* if there exists a permutation on the set N, which transforms one system into another. It is well known that $STS(n)$ exists if and only if $n \equiv 1$ or $3 \pmod 6$. Therefore for an orientable case we have $n \equiv 3$ or $7 \pmod{12}$ and it is known (see [2, 7]) that if a tiling of order n of an orientable sphere exists, the surface should be a sphere with $(n-4)(n-3)/12$ handles. For a nonorientable case $n \equiv 1$ or $3 \pmod 6$, $n > 7$, and therefore a tiling of a sphere with $(n-4)(n-3)/6$ crosscaps should exist.

From design theoretic point of view one can forget about topological aspects and get the following interesting equivalent combinatorial problem. To find a pair of $STS's$ of order n tiles a surface in black and white tiles one has to construct a pair of $STS's$ defined on the same set N and satisfy the following property:
each point $i \in N$ has *cyclic property* which means that all triangles containing i can be placed on a plane around the point i in the following cyclic way

$$i|i_1, i_2|i_3, i_4| \ldots |i_{n-2}, i_{n-1}|,$$

here vertical lines are put to share triples of these two $STS's$. It means that the following triples

$$(i, i_1, i_2),\ (i, i_3, i_4), \ldots, (i, i_{n-2}, i_{n-1})$$

are included in the first $STS(n)$ and triples

$$(i, i_2, i_3),\ (i, i_4, i_5), \ldots, (i, i_{n-1}, i_1)$$

are from the second one. As a consequence these $STS's$ are not intersected by blocks.

Let us consider a short survey of results on tilings of order n (or what is the same face 2-colorable embeddings of K_n). First in 1970 Youngs [9] showed that for $n \equiv 7 \pmod{12}$ there existed such tiling of an orientable surface. From Ringel's book 1974, see [2], one can conclude that for $n \equiv 3 \pmod{12}$ there exists a tiling of order n of an orientable surface given by well known Bose construction [8] and its isomorphic copy, and for $n \equiv 3 \pmod 6$, $n \geq 9$, there exists a tiling of a nonorientable surface. In 1978 Ducrocq and Sterboul [10] found for $n \equiv 3 \pmod 6$, $n \geq 9$, a tiling of order n of a nonorientable surface given by Bose construction [8] and its isomorphic copy. Not all cases of embeddings of a complete graph K_n described in Ringel book give tilings. After 20 years the following new results appeared. In 1998 Grannel, Griggs and Širáň [4, 5] proved that it is possible to construct a tiling of order $3n-2$, where $n \equiv 3$ or $7 \pmod{12}$ using as input a tiling of order n of an orientable surface. The same authors, see [4], established that there exist nonequivalent embeddings of K_n in a nonorientable surface for half of the residue class $n \equiv 1 \pmod 6$, $n > 7$, one of them is a tiling. In 2000 Bonnington, Grannel, Griggs and Širáň [6] found at least $2^{\frac{n^2}{54}-0(n)}$ tilings of order n of an orientable surface for $n \equiv 7$ or $19 \pmod{36}$. They also get a similar estimation for orientable case if $n \equiv 19$ or $55 \pmod{108}$, for nonorientable case if $n \equiv 1$ or $7 \pmod{18}$ and an improved estimation if $n \equiv 1$ or $19 \pmod{54}$. For information about tilings of small orders see [5, 11].

2 Constructions

In the section we are going to present a class of pairs of STS's of any order $n \equiv 3 \pmod 6$ which give us tilings of a nonorientable closed surface. The base construction is a special type of equivalent copy of well known Bose construction of STS. We will use the same notations as in known Hall book [8], see chapter 15, pages 230-233. The equivalent copy of Bose's STS of order $n \equiv 3 \pmod 6$ with an automorphism group, which contains an Abelian group A additively written as a subgroup is presented below in Theorem 1. The group A is an additive group of residues modulo $2t+1$, so we have 3 orbits of elements, $n = 6t+3$, where 0 is the zero of A and elements of A are $\{0, 1, \ldots, 2t\}$. Each i'th element of the j'th orbit will be labeled as i_j. The difference between elements from the same orbit is called a *pure* difference and from different orbits is called a *mixed* difference. We will labeled base block in orbit blocks by square brackets. Using the group A every such block generates $2t+1$ blocks in the STS(n). Other blocks in the system will be labeled by round brackets.

Theorem 1. *(See [8] and also [12].) With the additive group of residues modulo $m = 2t+1$, the blocks*

$$\begin{gathered} [1_1, (2t)_1, l_2], \ldots, [t_1, (t+1)_1, l_2], \\ [1_2, (2t)_2, m_3], \ldots, [t_2, (t+1)_2, m_3], \\ [1_3, (2t)_3, (-l-m)_1], \ldots, [t_3, (t+1)_3, (-l-m)_1], \\ [0_1, l_2, (l+m)_3], \end{gathered} \tag{1}$$

form a base for $STS(n)$, where $n = 6t+3$ and l, m are any numbers from $\{0, 1, \ldots, 2t\}$.

Proof. Let us show that the set of blocks (1) defines $STS(n)$, where $n = 6t+3$. Denote the system by S_n. It is easy to check that

a) every block contains three elements;

b) the number of elements equals to $n = 3(2t+1) = 6t+3$.

c) Let us count the number of blocks in the system S_n. By the construction, see (1), the number of base blocks equals to $3t+1$. Because the order of the automorphism group A is $2t+1$ every base block generates $2t+1$ blocks. As the result one gets

$$(2t+1)(3t+1) = n(n-1)/6$$

blocks in the system S_n.

d) It is not difficult to check from (1) that every element from N appears in $(n-1)/2$ triples.

e) At the end of the proof of the theorem we have to clarify if every pair of elements from N is only in one triple. Without loss of generality it is sufficient to consider the first t basis blocks

$$[i_1, j_1, l_2] \tag{2}$$

from (1), where

$$i = 1, \ldots, t,\ i + j \equiv 0 \pmod{(2t+1)}. \tag{3}$$

To get a pure difference d of the first element orbits not equal to 0 by modulo $2t+1$ it is necessary to have

$$i - j \equiv d \pmod{(2t+1)}.$$

From this equation and (3) we have

$$2i \equiv d \pmod{(2t+1)}.$$

Hence i and j are uniquely determined. Blocks (2) give every mixed difference of the first and the second element orbits with the exception l. The block

$$[0_1, l_2, (l+m)_3]$$

from (1) gives the mixed difference l. Therefore the set S_n is $STS(n)$. X-Mozilla-Status: 0000 X-Mozilla-Status2: 00000000

Corollary 1. *If $l = m = 0$ the $STS(n)$ from Theorem 1 is Bose [8] triple system.*

Denote further the Bose system by $STS(B, n)$.

Theorem 2. *The triple system $STS(B,n) \cup S_n$, where S_n is a STS from Theorem 1 and l, m are any numbers such that $(l, 2t{+}1) = 1$, $(m, 2t{+}1) = 1$, $(l{+}m, 2t{+}1) = 1$, $l \neq 0$, $m \neq 0$, gives a tiling of order n of a closed surface.*

Proof. Without loss of generality it is sufficient to consider any element from any orbit, for example the element 0_1, and prove that this element possesses cyclic property defined above. It means that all triples from $STS(B,n)$ and S_n can be placed around the point 0_1 in the following cyclic way

$$0_1|0_2, 0_3|x_3^1, x_3^2| \ldots |x_3^{2t-1}, x_3^{2t}|y_2^1, z_1^1| \ldots |y_2^{2t}, z_1^{2t}|, \tag{4}$$

where triples

$$(0_1, 0_2, 0_3),\ (0_1, x_3^1, x_3^2), \ldots, (0_1, x_3^{2t-1}, x_3^{2t}), (0_1, y_2^1, z_1^1), \ldots, (0_1, y_2^{2t}, z_1^{2t})$$

belong to $STS(B,n)$ and other triples $(0_1, 0_3, x_3^1), \ldots, (0_1, z_1^{2t}, 0_2)$ are from the system S_n. To show it we are going to find an explicit form of this row (4). First consider an element $x_3^{2i-1}, i = 1, \ldots, t$. According to the structure of the system S_n, see the third row in (1), base blocks containing the element $(-l-m)_1$ of the first orbit contain elements from the third orbit. Therefore the element $(-l-m)_1$ meets in blocks with $2t$ nonzero elements of the third orbit. The element $(-l-m)_1$ meets zero element in one of the cyclic switchings of the last base block from (1). Then between these elements there is the element $(-l-m)_3$. Let this base triple in S_n have the following presentation

$$[x_3, (-l-m)_3, (-l-m)_1],$$

where according to (1), $x - l - m \equiv 0 \pmod{2t+1}$ is true and therefore $x \equiv l+m \pmod{2t+1}$. Hence we have the base block

$$[(l+m)_3, (-l-m)_3, (-l-m)_1)] \in S_n$$

and then its cyclic switching by the element $l+m$ gives the block

$$((2(l+m))_3, 0_3, 0_1) \in S_n,$$

hence $x_3^1 = (2(l+m))_3$.

By the construction of the system $STS(B,n)$ we have the base block

$$[i_3, (2t+1-i)_3, 0_1] \in STS(B,n),$$

then $((2(l+m))_3, (-2(l+m))_3, 0_1) \in STS(B,n)$ and $x_3^2 = (-2(l+m))_3$. The block

$$[(-l-m+i)_3, (l+m-i)_3, (-l-m)_1]$$

is the base block in S_n according to (1). Therefore its cyclic switching by the element $l+m$ gives the block

$$(0_1, i_3, (2(l+m)-i)_3)) \in S_n \tag{5}$$

for any $i = 1, \ldots, t$. For the system $STS(B,n)$ we also have base blocks

$$[0_1, (-i)_3, i_3)], [0_1, (2(l+m)-i)_3), (-2(l+m)+i)_3], \tag{6}$$

surrounding the block (5) in the cyclic presentation (4). From (6) one can get

$$x_3^{2s+1} - x_3^{2s-1} = 2(l+m) - i - (-i) = 2(l+m)$$

for any $s = 1, \ldots, t$. Therefore elements x_3^{2s-1} from (4) define the following arithmetical progression

$$2(l+m), 4(l+m), \ldots, 2t(l+m),$$

where according to the condition of this theorem $(l+m, 2t+1) = 1$ and then $x_3^i \neq x_3^j$ for any $i \neq j$, where $i, j \in \{1, 2, \ldots, 2t.\}$ X-Mozilla-Status: 0000 X-Mozilla-Status2: 00000000

The triple

$$[0_1, 2t(l+m))_3, (-2t(l+m))_3] \in STS(B,n).$$

Using the equality $-2t(l+m) \equiv (l+m) \pmod{2t+1}$ and the block

$$[0_1, l_2, (l+m)_3] \in S_n$$

(see (1)) we get $y_2^1 = l_2$ in (4). Similar considerations show that elements

$$y_2^1, y_2^2, \ldots, y_2^{2t}$$

define an arithmetical progression with the difference l and elements

$$z_1^1, z_1^2, \ldots, z_1^{2t}$$

an arithmetical progression with the difference $2l$ and with the starting element $(2l)_1$.

Therefore $y_2^{2t} = (2tl)_2$, $z_1^{2t} = (4tl)_1$ and the triple $(0_1, (2tl)_2, (4tl)_1)$ have to belong to the system $STS(B, n)$. Let us prove it. Using the congruence

$$4t \equiv (2t-1) \pmod{(2t+1)}, \tag{7}$$

we get for the triple the following presentation

$$(0_1, (2tl)_2, (4tl)_1) = (0_1, (2tl)_2, ((2t-1)l)_1). \tag{8}$$

The cyclic switch of the last triple by the element l gives the base triple

$$[l_1, (2tl+l)_2, (2tl-l+l)_1] = [l_1, 0_2, (2tl)_1],$$

from the system $STS(B, n)$. Therefore the triple (8) also belongs to the system $STS(B, n)$. The triple $[l_2, (2tl)_1, (-2tl)_1]$ is a base triple from the system S_n, therefore its cyclic switch by the element $2tl$ gives using (7) the triple

$$(((2t+1)l)_2, (4tl)_1, 0_1) = (0_2, ((2t-1)l)_1, 0_1) \in S_n.$$

So we have enumerated all $n-1$ triples of systems $STS(B, n)$ and S_n containing the element 0_1. Analogous considerations can be done for elements 0_2 and 0_3, for example for the element 0_3 we have the presentation

$$0_3|0_1, 0_2|u_2^1, u_2^2| \ldots |u_2^{2t-1}, u_2^{2t}|v_1^1, w_3^1| \ldots |v_1^{2t}, w_3^{2t}|. \tag{9}$$

The proof is done.

It is well known from the main Theorem of surface topology, see, for example [7], that two closed surfaces are homeomorphic if and only if both of them have the same Euler characteristic and both of them are either oriented or nonoriented. If Euler characteristic is odd then the surface is nonorientable. In our case $n \equiv 3 \pmod 6$. It is easy to calculate that for $n \equiv 9 \pmod{12}$ the Euler characteristic is odd and the system $STS(S, n) \cup S_n$ gives a tiling of a sphere with $(n-3)(n-4)/6$ crosscaps. In the case $n \equiv 3 \pmod{12}$ after glueing together all triangles corresponding to all triples of the system $STS(S, n) \cup S_n$ in order to get a topological polygon and investigating a boundary of this resulting polygon one can find at least two edges with the same orientation. It means that we also obtain a tiling of a nonorientable surface. So we get the following statement.

Corollary 2. *Any triple system $STS(B, n) \cup S_n$ from Theorem 2 gives a tiling of a nonorientable closed surface (a sphere with $(n-3)(n-4)/6$ crosscaps).*

Using different admissible l, m satisfying to Theorem 2 we get a large class of triple systems of order n giving tilings. Some of such triple systems may be or may not be equivalent.

Taking as input any tiling of order n from Theorem 2 and Corollary 2 and use the same approach as Grannel, Griggs and Širáň applied in [4] we get the following result.

Theorem 3. *There exists a class of tilings of a nonorientable closed surface (a sphere with $(n-3)(n-4)/6$ crosscaps) by STS's of order n for half of the residue class $n \equiv 1 \pmod 6$.*

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